WADC TECHNICAL REPORT 52-164

SHOCK REFLECTION FROM EDGES AND FROM SLOTTED WALLS

T. F. SUN
CORNELL UNIVERSITY

JULY 1952

WRIGHT AIR DEVELOPMENT CENTER

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FOREWORD

This report is submitted as part of Contract AF 33 (038) 21406, which was initiated by the Office of Air Research. Work under this contract was begun at the Graduate School of Aeronautical Engineering at Cornell University in March 1951. The report is one of a series to be published as the result of the work carried out under this contract.

The present investigation was carried out partly under the support of the contract mentioned above and partly as the author's thesis investigation for the degree Master of Aeronautical Engineering. The subject of the investigation was suggested to the author by Professor J. M. Wild, now Acting Chief Engineer of the ARO Corporation. The author wishes to acknowledge his indebtedness to Mr. Wild and to the Faculty of the Graduate School of Aeronautical Engineering of Cornell University and in particular to Professor W. R. Sears for his suggestion of this study and his invaluable advice and encouragement throughout the study and to Professor N. Rott for his most helpful suggestions and criticism while preparing this report.

The work on this project was conducted under Research and Development Order No. 465-5-6, Practical Problems in Aerodynamics. Mr. L. S. Wasserman of the Flight Research Laboratory, Wright Air Development Center, was the project engineer

ABSTRACT

The flow characteristics behind weak stationary shock waves reflected from various edges lying in the mainstream direction are determined according to linearized theory. Five different types of edges are considered, made up of various combinations of solid and free plane surfaces. An assumption regarding the singularity of one of the perturbation velocity is required in order to render the solutions unique.

By superposition of such basic "edge" solutions, the flow behind a shock wave reflected from a wall with a slot (i.e., a strip of free-surface between panels of solid wall) and from multiply-slotted walls are obtained. These solutions apply only to regions upstream of multiple interactions of the slot edges; however, these regions include the most interesting regions of flow in the case of reflection from slotted wind-tunnel wall, for example.

The relation of the single-slot problem to the problem of a narrow rectangular supersonic wing is discussed.

PUBLICATION REVIEW

FOR THE COMMANDING GENERAL:

Harry C. Henry for CP

LESLIE B. WILLIAMS, Colonel, USAF
Chief, Flight Research Laboratory
Research Division

CONTENTS

INTRODUCTION

In recent supersonic wind tunnel research work, there occurs sometimes the necessity of a slot, or multiple slots, along the test section, parallel to the stream. The slot is usually in the form of a strip of free surface, say, stationary air, between neighbouring solid walls. Also there may be mutually perpendicular edges in various combinations of solid wall and free surface. When a two-dimensional shock wave from a wind-tunnel model is reflected from the slot or the edge, the flow characteristics after reflection becomes three-dimensional. The determination of the flow field is the subject of the present study.

In Part I of this thesis, basic ideas and equations are discussed and derived. In Part II, flow characteristics after a weak shock hitting an edge are determined. (Five different kinds of edge problems are considered). Then the problems of a shock reflected from a single slot in infinite solid wall and from a multiply-slotted solid wall are considered in Part III and IV respectively. In Part V, correspondence of boundary conditions for velocity component w of the narrow rectangular supersonic wing and u of the narrow slot problem are established. It is shown then that their difference in singularities makes the solutions of the two problems not identical.

PART I

BASIC CONSIDERATIONS

Let U be the undisturbed supersonic velocity along x axis. When the stream passes over some boundaries or obstacles, there is velocity disturbance which may be denoted by its components u, v, w along x, y, z axes respectively. Assume the flow to be irrotational and steady and denote ϕ as the perturbation velocity potential such that

$$u = \frac{\partial \phi}{\partial x} \quad , \quad v = \frac{\partial \phi}{\partial y} \quad , \quad w = \frac{\partial \phi}{\partial z} \quad .$$

These perturbation velocity components as well as perturbation density, pressure are supposed so small, compared with the undisturbed values, that their square terms or cross product can be neglected. Using Euler's equations of motion, continuity equation, together with the equation of state of gas, assumed perfect, one can easily derive the well-known Prandtl-Glauert equation

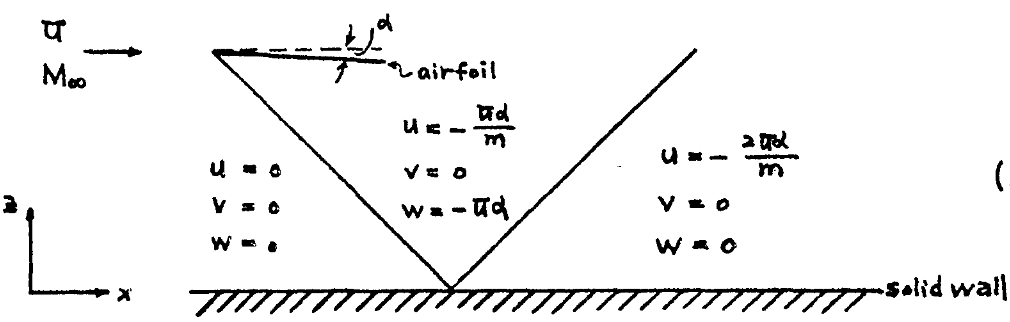
$$m^2 \Omega_{xx} - \Omega_{yy} - \Omega_{zz} = 0 \tag{1.1}$$

Where Ω denotes ϕ , u, v, w or perturbation pressure, and $m^2 = M_\infty^2 - 1$, M_∞ being the free stream Mach number.

Before going into the edge problems, it will be advisable to present first the two dimensional result of

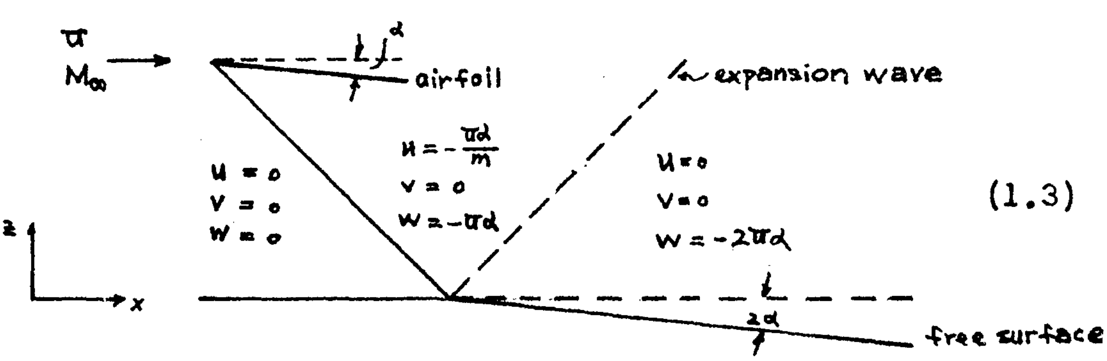
a shock reflected from a solid surface or from a free surface.

Assume a weak compression shock, such as that results from a supersonic flow about a thin airfoil at a small angle of attack α, impinges on a solid wall, infinite in extent. A compression shock will be reflected from the wall. Consistent with the assumption of small perturbation, the angle of reflection will be same as angle of incidence which should be taken to be equal to Mach angle $\sin^{-1}\frac{1}{M_\infty}$ or $\tan^{-1}\frac{1}{m}$ in the present approximation. Values of perturbed velocity components are summarized in the following figure:



(1.2)

If a weak compression shock impinges on a free surface, an expansion wave will result after reflection. Perturbed velocity components has the following values:

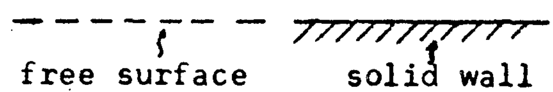


(1.3)

Again, within the present approximation, both angle of incidence and angle of reflection will be equal to Mach Angle. Moreover, although the free surface will deflect downward through an angle of 2α after reflection, we will still consider it undeflected in the edge problems discussed.

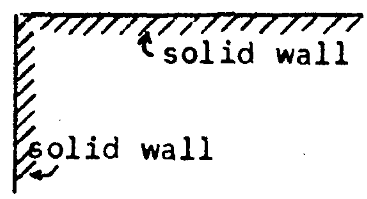
Five different kinds of edges are going to be considered in Part II:

(A)



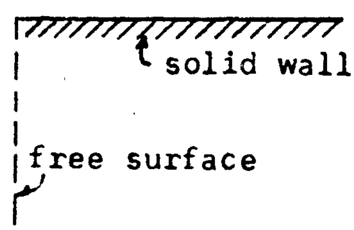
(1.4)

(B)



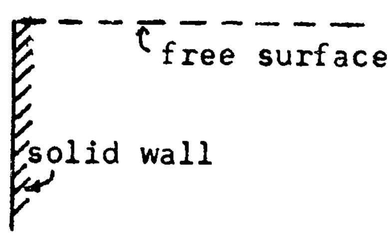
(1.5)

(C)



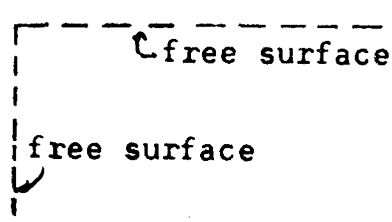
(1.6)

(D)



(1.7)

(E)



(1.8)

When a shock hits an edge, the flow problem around the edge becomes a three dimensional one. However, due to the principle of forbidden signals of supersonic aerodynamics, (Reference 1), the edge effect is restricted within the Mach cone which has an apex at the intersection point of the shock and the edge. Outside Mach cone, the flow characteristics are still two dimensional as mentioned before. Furthermore, the present problems contain no characteristic length. By dimensional theory argument (Reference 2), one concludes that the edge effect results in a conical problem, i.e. all flow characteristics are constant along the ray radiating from the apex of Mach cone. The powerful technique of conical flow may then be employed here.

Change cartesian coordinates x, y, z to cylindrical one x, ϖ, ω where

$$\begin{aligned} \varpi &= \sqrt{y^2 + z^2} \\ \omega &= \tan^{-1} \frac{z}{y} \end{aligned} \qquad (1.9)$$

Equation (1.1) reduces to

$$m^2 \Omega_{xx} - \Omega_{\varpi\varpi} - \frac{1}{\varpi} \Omega_{\varpi} - \frac{1}{\varpi^2} \Omega_{\omega\omega} = 0 \qquad (1.10)$$

If Ω denotes the cartesian velocity components u, v, w or perturbation pressure, <u>but not velocity potential</u>, ϕ,

conical flow properties ensure that Ω will be a function of $\frac{r}{x}$, ω only. By introducing

$$\eta = \frac{r}{x} \tag{1.11}$$

equ. (1.10) becomes

$$(m^2\eta^2 - 1)\,\Omega_{\eta\eta} + (2m^2\eta - \tfrac{1}{\eta})\,\Omega_{\eta} - \frac{1}{\eta^2}\,\Omega_{\omega\omega} = 0 \tag{1.12}$$

Use Tschaplygin transformation (Reference 2)

$$s = \frac{m\eta}{1 + \sqrt{1 - m^2\eta^2}} \qquad \text{or} \qquad \eta = \frac{2}{m} \cdot \frac{s}{1+s^2} \tag{1.13}$$

(Note that on Mach cone: $\eta = 1/m$, $s = 1$)

(1.12) reduces to

$$\Omega_{ss} + \frac{1}{s}\,\Omega_s + \frac{1}{s^2}\,\Omega_{\omega\omega} = 0 \tag{1.14}$$

which is the Laplace equation of Ω in s, ω

To find Ω = u, v, w within the Mach cone, one has to solve equation (1.14) with the appropriate boundary values of Ω or $\frac{\partial \Omega}{\partial n}$ (the normal derivative) described completely along the boundary (some of these boundary values are immediately known, some would be determined with aids of irrotationality). Thus we usually have the so called "mixed boundary value problems." Technique to find the solution and uniqueness of solution and singularity behaviors will be considered in Part II.

Note that if one of the velocity components is known,

the other two may be found by irrotationality condition, e.g. if u is known, v, w may be obtained from

$$v = \int \left[\frac{\sin\omega}{\eta^2} \frac{\partial u}{\partial \omega} - \frac{\cos\omega}{\eta} \frac{\partial u}{\partial \eta} \right] d\eta + F(\omega) \qquad (1.15)$$

$$w = -\int \left[\frac{\sin\omega}{\eta} \frac{\partial u}{\partial \eta} + \frac{\cos\omega}{\eta^2} \frac{\partial u}{\partial \omega} \right] d\eta + G(\omega) \qquad (1.16)$$

or in (s, ω)

$$v = \frac{m}{2} \int \left[\frac{1-s^2}{s^2} \sin\omega \frac{\partial u}{\partial \omega} - \frac{1+s^2}{s} \cos\omega \frac{\partial u}{\partial s} \right] ds + F(\omega) \qquad (1.17)$$

$$w = -\frac{m}{2} \int \left[\frac{1-s^2}{s^2} \cos\omega \frac{\partial u}{\partial \omega} + \frac{1+s^2}{s} \sin\omega \frac{\partial u}{\partial s} \right] ds + G(\omega) \qquad (1.18)$$

Usually one is interested in the pressure distribution. As is well-known in the perturbation theory, the pressure coefficient c_p is given by

$$c_p \equiv \frac{p - p_\infty}{\frac{1}{2} \rho_\infty U_\infty^2} = -2 \frac{u}{U} \qquad (1.19)$$

PART II

SHOCK REFLECTED FROM AN EDGE

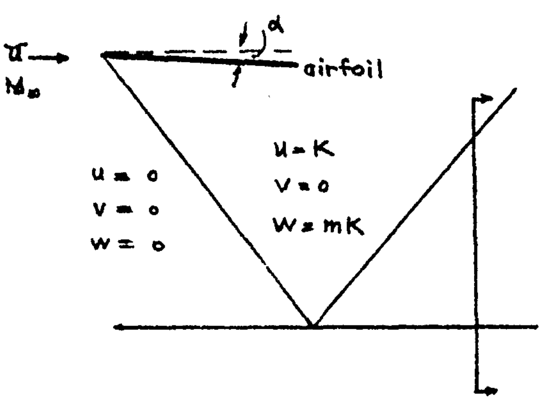
We shall consider the flow characteristics of the edge problems for the five cases mentioned on page 4 one by one. In case A, we shall develope the method solving the problem in detail, and find the three velocity components u, v, W completely. In other cases, only u (i.e. the pressure) is solved; v and W can always be found by means of (1.15) -- (1.18) if required.

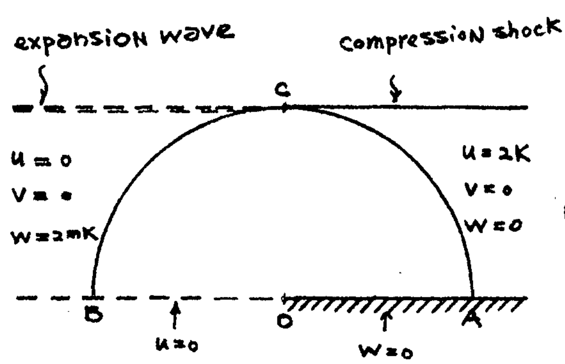
The notation

$$K \equiv -\frac{U\alpha}{m}$$
$$\text{or } mK = -U\alpha$$

is used in this part.

Case A:





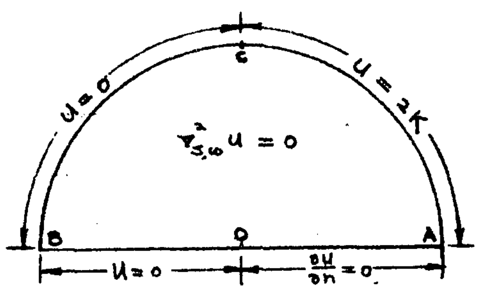
(A.1

Let us find u first. The boundary values of u along Ac is 2K and along BC and Bo, zero. Along oA, we only

have W=o. However, as there is no change of W along x direction, we have $\frac{\partial W}{\partial x} = 0$ and hence by irrotationality $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial n} = 0$ along OA. Thus the problem of finding u due to edge effect is to solve

$$\nabla^2_{s,\omega} u \equiv \frac{\partial^2 u}{\partial s^2} + \frac{1}{s}\frac{\partial u}{\partial s} + \frac{1}{s^2}\frac{\partial^2 u}{\partial \omega^2} = 0 \tag{A.2}$$

in the semi-circular region subjected to boundary conditions as shown in Fig. (A.3):



(A.3)

To solve (A.3), use conformal transformation

$$\zeta = \sigma^{\frac{1}{2}} \tag{A.4}$$

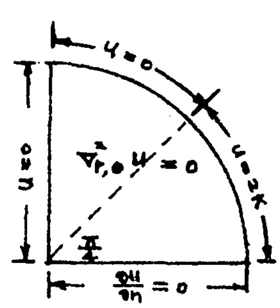
where

$$\begin{cases} \zeta = re^{i\theta} \\ \sigma = se^{i\omega} \end{cases} \tag{A.5}$$

or

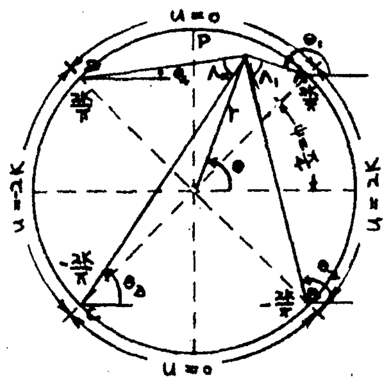
$$\begin{cases} s = r^2 \\ \omega = \frac{2\theta}{\pi} \end{cases} \qquad \begin{cases} r = s^{\frac{1}{2}} \\ \theta = \frac{\omega}{2} \end{cases} \tag{A.6}$$

The problem is transformed to



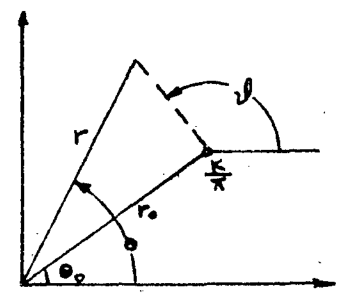
(A.7)

As the sides with $u = o$ and $\frac{\partial u}{\partial n} = o$ requires antisymmetric and symmetric continuation respectively, Fig. (A.7) may be completed to a full circle:



(A.8)

One of the fundamental solution of Equation (A.2) is $u = \frac{K}{\pi}\vartheta$, where ϑ is the angle shown in the Figure. This may be verified by direct substitution and may be imagined as the potential of a vortex of strength $\frac{K}{\pi}$ situated at r_o, θ_o. To solve the problem (A.8), one may place vortices of strength $+\frac{2K}{\pi}$, $-\frac{2K}{\pi}$, $-\frac{2K}{\pi}$ and $+\frac{2K}{\pi}$ at A, B, c, D respectively. Then at any point r, θ inside the circle,



$$u = \frac{2K}{\pi}\theta_1 - \frac{2K}{\pi}\theta_2 - \frac{2K}{\pi}\theta_3 + \frac{2K}{\pi}\theta_4 + \text{constant}$$

$$= \frac{2K}{\pi}(\theta_1 - \theta_2) - \frac{2K}{\pi}(\theta_3 - \theta_4) + \text{constant}$$

$$= \frac{2K}{\pi}(\Lambda_1 - \Lambda_2) + \text{constant}$$

To determine the arbitrary constant, take point p. Here $\Lambda_1 = \Lambda_2 = \frac{\pi}{4}$, and $u = o$ as given. Hence the constant has to vanish.

$$u = \frac{2K}{\pi}(\Lambda_1 - \Lambda_2) \tag{A.9}$$

It can be easily seen that the boundary conditions along AB, BC, CD are all satisfied. For example, along AB, Λ_1 and Λ_2 always take constant values $\frac{5\pi}{4}$ and $\frac{\pi}{4}$ respectively. Then u = 2K as required.

To express Λ_1, Λ_2 in terms of r, θ , remember that the circle is an unit circle. By some algebric manipulation, we have

$$\tan\Lambda_1 = \frac{2\sin\psi(\cos\psi - r\cos\theta)}{\cos 2\psi - 2r\cos\psi\cos\theta + r^2} \tag{A.10}$$

$$\tan\Lambda_2 = \frac{2\sin\psi(\cos\psi + r\cos\theta)}{\cos 2\psi + 2r\cos\theta\cos\psi + r^2} \tag{A.11}$$

and $$\Lambda \equiv \Lambda_1 - \Lambda_2$$

$$= \tan^{-1}\left\{\frac{4\sin\psi\, r(1-r^2)\cos\theta}{r^4 + 2r^2(\cos 2\psi - 2\cos^2\theta) + 1}\right\} \tag{A.12}$$

Here $\psi = \frac{\pi}{4}$

$$\Lambda = \tan^{-1}\left\{\frac{2\sqrt{2}\, r(1-r^2)\cos\theta}{r^4 - 4r^2\cos^2\theta + 1}\right\} \tag{A.13}$$

By means of (A.6)

$$\Lambda = \tan^{-1}\left\{\frac{2\sqrt{2}\cos\frac{\omega}{2}\, s^{\frac{1}{2}}(1-s)}{s^2 - 4s\cos^2\frac{\omega}{2} + 1}\right\} \tag{A.14}$$

Finally, by equ. (1.13) the solution may be expressed in physical coordinates η , ω:

$$U = -\frac{2\pi\alpha}{m} \cdot \frac{\Lambda}{\pi} \tag{A.15}$$

where

$$\Lambda = \tan^{-1}\left\{ \frac{2\sqrt{m\eta(1-m\eta)}\ \cos\frac{\omega}{2}}{1-2m\eta\cos^2\frac{\omega}{2}} \right\} \tag{A.16}$$

and

$$\begin{aligned} &0 \le \Lambda < \pi && \text{when} \quad 0 \le \omega \le \pi, \ m\eta < 1 \\ &\Lambda = \pi && \text{when} \quad 0 \le \omega \le \tfrac{\pi}{2}, \ m\eta = 1 \\ &\Lambda = 0 && \text{when} \quad \tfrac{\pi}{2} \le \omega \le \pi, \ m\eta = 1 \end{aligned} \tag{A.17}$$

The pressure coefficient is, due to (1.19)

$$c_p = \frac{4\alpha}{m} \cdot \frac{\Lambda}{\pi} \tag{A.18}$$

where Λ takes same value as in (A.16), (A.17).

On free surface: $\omega = \pi$, $\Lambda = 0$

$$c_p = 0$$

on solid surface: $\omega = 0$, $\Lambda = \cos^{-1}(1-2m\eta)$

$$c_p = \frac{4\alpha}{m\pi} \cos^{-1}(1-2m\eta) \tag{A.19}$$

The distribution of c_p along these surface and a set of constant pressure lines (iso-bar) has been plotted in Figure 1 and 2 (P. 47, 48) respectively.

The point c in Figure 2, p. 48 is quite interesting. At this point the boundary values given by the two-dimensional results are discontinuous, and the expression of (A.16) takes the indeterminate form $\tan^{-1}(\frac{0}{0})$. It follows that every iso-bar of different values of c_p

converges to the point. Moreover, the tangents of all iso-bars are horizontal (parallel to boundary surface) at c except the one of $c_p = 2\frac{\alpha}{m}$, which makes 135° with the horizontal (see Appendix I).

The position variable used in Figure 2 is $m\eta$. However, if we assume m = 1 (i.e. $M_\infty = \sqrt{2}$) and give x a definite value, say 1, $m\eta$ represents the actual physical radial distance ϖ. It is interesting to note that: when going inward radially from points along AC the pressure decreases quite rapidly, and from points along BC the pressure increases very rapidly. The phenomena becomes more acute in the neighborhood of C.

These statements may be clearly seen from the following expression of $\frac{\partial c_p}{\partial \varpi}$

$$\frac{\partial c_p}{\partial \varpi} = \frac{4\alpha}{\pi} \cdot \frac{\cos\frac{\omega}{2}\left(1 - 2m\eta \sin^2\frac{\omega}{2}\right)}{x\sqrt{m\eta(1-m\eta)}\ \left(1 - m^2\eta^2 \sin^2\omega\right)} \qquad (A.20)$$

Here, we find

$$\frac{\partial c_p}{\partial \varpi} > 0 \quad \text{if} \quad 1 - 2m\eta \sin^2\frac{\omega}{2} > 0\ , \ \text{i.e.}\ \sin\frac{\omega}{2} < \frac{1}{\sqrt{2m\eta}}$$

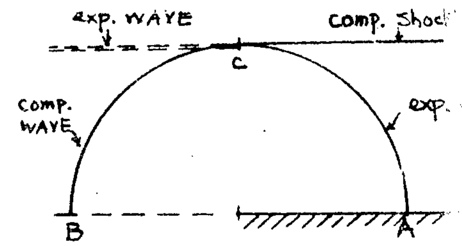
$$\frac{\partial c_p}{\partial \varpi} < 0 \quad \text{if} \quad 1 - 2m\eta \sin^2\frac{\omega}{2} < 0\ , \ \text{i.e.}\ \sin\frac{\omega}{2} > \frac{1}{\sqrt{2m\eta}}$$

and when $m\eta = 1$

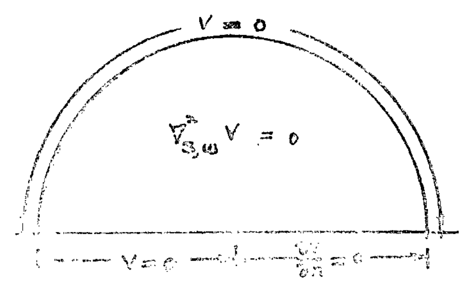
$$\frac{\partial c_p}{\partial \varpi} = +\infty \qquad \text{for} \quad 0 \leq \omega < \frac{\pi}{2}$$

$$\frac{\partial c_p}{\partial \varpi} = -\infty \qquad \text{for} \quad \frac{\pi}{2} < \omega < \pi$$

From these considerations, we are easily led to assert that AC and BC are expansion wave and compression wave respectively. They may be considered as the continuation of the originally reflected waves, as shown in the accompanied figure.



Now let us consider V. On Figure (A.1), along AC and BC, $V = 0$. Along oA, $\frac{\partial w}{\partial y} = 0$ gives $\frac{\partial v}{\partial z} = \frac{\partial v}{\partial n} = 0$. Along BO, $\frac{\partial u}{\partial y} = 0$. Hence $\frac{\partial v}{\partial x} = 0$ by irrotationality i.e. V is constant along x. Since $V = 0$ on Mach cone (bounded by 2-dim. flow), this constant vanishes, and we have $V = 0$ on the entire free surface. The problem of solving V is to solve $\nabla^2_{3,\omega} V = 0$ in the following region.



(A.2)

By the well-known proof of uniqueness of solution of Dirichlet's and Neumann's Problem, the solution of (A.2) will be identically equal to zero, if there is no singularity within or on the boundary (reference 3). However, we don't know if there is any singularity at the present time. Let's find v by equation (1.15) (irrotationality condition).

By (A.15), (A.16)

$$\frac{\partial u}{\partial \eta} = - \frac{2\pi\alpha \cos\frac{\omega}{2}\left(1 - 2m\eta \sin^2\frac{\omega}{2}\right)}{\pi\sqrt{m\eta(1-m\eta)}\left(1-m^2\eta^2\sin^2\omega\right)} \tag{A.22}$$

$$\frac{\partial u}{\partial \omega} = \frac{2\pi\alpha \sin\frac{\omega}{2}\left(1 + 2m\eta \cos^2\frac{\omega}{2}\right)\sqrt{m\eta(1-m\eta)}}{m\pi\left(1-m^2\eta^2\sin^2\omega\right)} \tag{A.23}$$

Due to (1.15)

$$\begin{aligned} V &= \frac{2\pi\alpha}{\pi}\cos\frac{\omega}{2}\int\frac{d(m\eta)}{(m\eta)^{\frac{3}{2}}(1-m\eta)^{\frac{1}{2}}} + F(\omega) \\ &= -\frac{4\pi\alpha}{\pi}\cos\frac{\omega}{2}\sqrt{\frac{1-m\eta}{m\eta}} + F(\omega) \end{aligned}$$

When $m\eta = 1$, $V = o$. Hence F(W) vanishes.

$$V = -\frac{4\pi\alpha}{\pi}\cos\frac{\omega}{2}\sqrt{\frac{1-m\eta}{m\eta}} \tag{A.24}$$

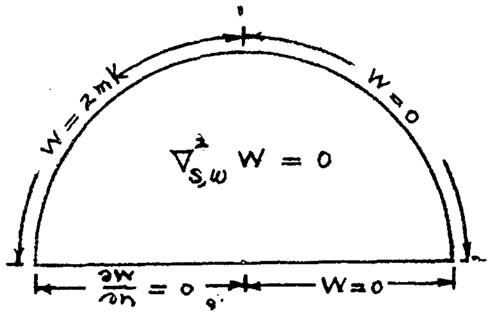
Thus V is not identically equal to zero, but has a singularity of $\frac{1}{\sqrt{m\eta}}$ type. This solution satisfies all boundary conditions of Fig. (A.21).

For W, refer to Fig. (A.1), we have W=o along oA, AC and W = $-2\pi\alpha$ = 2 Km along CB. Along Bo, since u = o , v = o, we have $\frac{\partial u}{\partial x} = o$, $\frac{\partial v}{\partial y} = o$.

By means of continuity equation in small perturbation theory

$$m^2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{A.25}$$

We have $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial n} = o$ along Bo. Thus to find W is to solve



(A.26)

The problem is similar to Fig. (A.3), to which it can be transformed. Replacing 2K in (A.9) by 2Km and ω in (A.16) by $(\pi - \omega)$, we have the solution for problem Fig. (A.26):

$$W = 2Km \frac{\Lambda}{\pi} = -2\pi\alpha \frac{\Lambda}{\pi} \tag{A.27}$$

where
$$\Lambda = \tan^{-1}\left\{ \frac{2\sqrt{2}\, s^{\frac{1}{2}} (1-s) \sin\frac{\omega}{2}}{s^2 - 4s \sin^2\frac{\omega}{2} + 1} \right\} \tag{A.28}$$

$$= \tan^{-1}\left\{ \frac{2\sqrt{m\eta(1-m\eta)}\, \sin\frac{\omega}{2}}{1 - 2m\eta \cdot \sin^2\frac{\omega}{2}} \right\} \tag{A.29}$$

and

$$\begin{aligned} 0 \le \Lambda < \pi \quad & \text{when} \quad 0 \le \omega \le \pi,\ m\eta < 1 \\ \Lambda = \pi \quad & \text{when} \quad \tfrac{\pi}{2} \le \omega \le \pi,\ m\eta = 1 \\ \Lambda = 0 \quad & \text{when} \quad 0 \le \omega \le \tfrac{\pi}{2},\ m\eta = 1 \end{aligned} \tag{A.30}$$

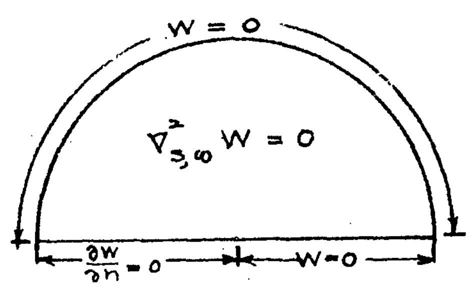
There may be also some singularities which satisfies zero boundary conditions (Note that Laplace eqn. is a linear differential equation and superposition of solutions may be used). Compute W by equ. (1.16). With aids of (A.22), (A.23) we have

$$W = \frac{2U\alpha}{\pi}\sin\frac{\omega}{2}\int\frac{(1-m\eta)+2m^2\eta^2\cos\omega\cos\frac{\omega}{2}}{(m\eta)^{\frac{3}{2}}(1-m\eta)^{\frac{1}{2}}(1-m^2\eta^2\sin^2\omega)}\,d(m\eta) + G(\omega)$$

$$= -\frac{2U\alpha}{\pi}\tan^{-1}\left\{\frac{2\sqrt{m\eta(1-m\eta)}\,\sin\frac{\omega}{2}}{1-2m\eta\sin^2\frac{\omega}{2}}\right\} - \frac{4U\alpha}{\pi}\sin\frac{\omega}{2}\sqrt{\frac{1-m\eta}{m\eta}} + G(\omega) \qquad (A.27_a$$

Where $G(\omega) = o$ if same value of $\tan^{-1}(\quad)$ as that in (A.30) is used. (Integration detail is given in Appendix II).

Here we see that in addition to the solution (A.27), we have a solution of singularity of $\frac{1}{\sqrt{m\eta}}$ type. The latter satisfies the following conditions:



(A.31)

By direct differentiation of v and W ($A.27_a$ and A.24), it can be easily verified that

$$\frac{\partial V}{\partial z} - \frac{\partial W}{\partial y} = 0$$

is also satisfied.

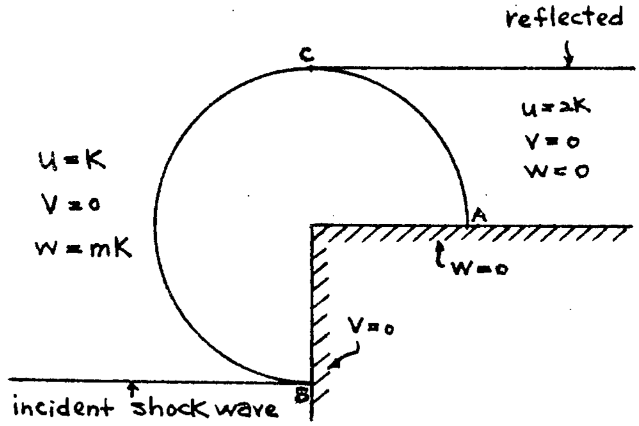
It is clear now that to solve the edge problem (and in fact many other problems in conical flow) one should be rather careful about the possible existence of a

singularity. It may be possible to set up the boundary value problems separately for u, v and w. However, the possible existence of singularity makes the solution of Laplace equation not unique even if the differential equation and the boundary conditions are properly satisfied. To make solution unique, it is necessary and sufficient to make physical assumptions on singularities for one of the quantities u, v, w. In our case, we shall assume there is no singularity in u (the assumption may be justified from a physical point of view) and find u uniquely by solving the mixed boundary-value problem. v and W admit singularities. They will be determined uniquely by means of irrotationality conditions, i.e., by equation (1.15), (1.16), if u is known.

It is also interesting to notice that in the present case, the regular part of W is a mirror image of u (compare A.27a with A.15) and the singular part is a mirror image of v (compare A.27a with A.24). That is,

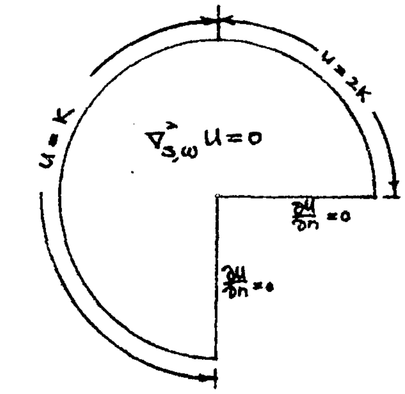
$$W(\eta, \omega) = m\, u(\eta, \pi-\omega) + v(\eta, \pi-\omega)$$

Case B:



(B.1)

Owing to irrotationality conditions and 2-dimensional values, the problem of finding u becomes that of solving the following boundary-value problems:

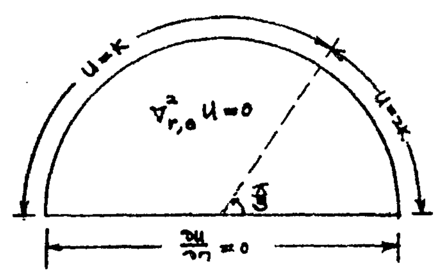


(B.2)

Use conformal transformation,

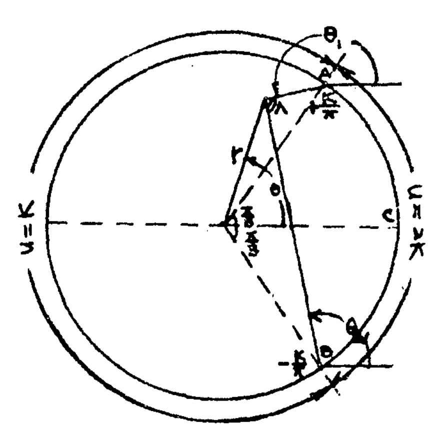
$$\zeta = \sigma^{\frac{2}{3}} \tag{B.3}$$

where $\zeta = re^{i\theta}$, $\sigma = se^{i\omega}$. The problem is then transformed to:



(B.4)

which has same solution as that of



(B.5)

To solve (B.5), place vortices $+\frac{K}{\pi}$ and $-\frac{K}{\pi}$ at A and B respectively. For any point (r, θ) inside the circle,

$$u = \frac{K}{\pi}(\theta_1 - \theta_2) + \text{constant}$$
$$= \frac{K}{\pi}\Lambda + \text{constant}$$

At point C, $\Lambda = \frac{4}{3}\pi$ and $u = 2K$ as given. Hence the constant should be equal to $\frac{2}{3}K$. We have

$$u = \frac{K}{\pi}\left(\Lambda + \frac{2}{3}\pi\right) \qquad (B.6)$$

To express Λ in r, θ, put $\psi = \frac{\pi}{3}$ in equ. (A.10)

$$\Lambda = \tan^{-1}\left\{\frac{\sqrt{3}\,(1 - 2r\cos\theta)}{2r^2 - 2r\cos\theta - 1}\right\} \qquad (B.7)$$

where

$$\begin{cases} \frac{\pi}{2} < \Lambda < \frac{3}{2}\pi & \text{when} \quad 0 \le \theta \le \frac{\pi}{3} \\ 0 < \Lambda < \pi & \text{when} \quad \frac{\pi}{3} \le \theta \le \pi \end{cases} \qquad (B.8)$$

Transform back to (s, ω)

$$\Lambda = \tan^{-1}\left\{\frac{\sqrt{3}\,(1 - 2s^{\frac{2}{3}}\cos\frac{2}{3}\omega)}{2s^{\frac{4}{3}} - 2s^{\frac{2}{3}}\cos\frac{2}{3}\omega - 1}\right\} \qquad (B.9)$$

where

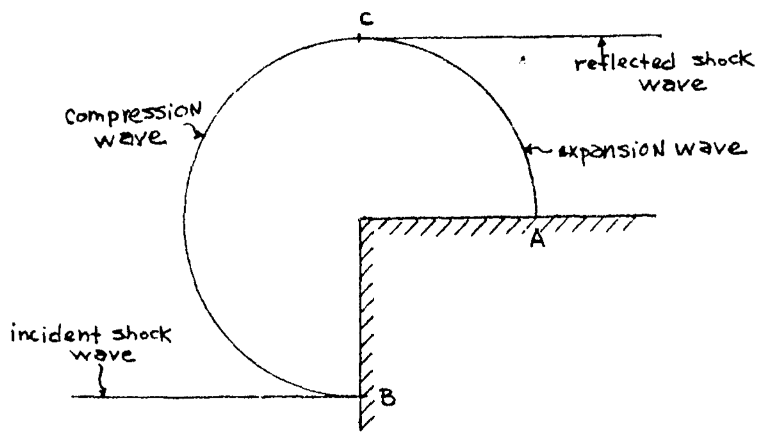
$$\begin{cases} \frac{\pi}{2} < \Lambda < \frac{3}{2}\pi & \text{when} \quad 0 \le \omega \le \frac{\pi}{2} \\ 0 < \Lambda < \pi & \text{when} \quad \frac{\pi}{2} \le \omega \le \frac{3}{2}\pi \end{cases} \qquad (B.10)$$

The pressure coefficient

$$c_p = \frac{2\alpha}{m\pi}\left(\Lambda + \frac{2}{3}\pi\right) \qquad (B.11)$$
$$= k_p \cdot \frac{\alpha}{m}$$

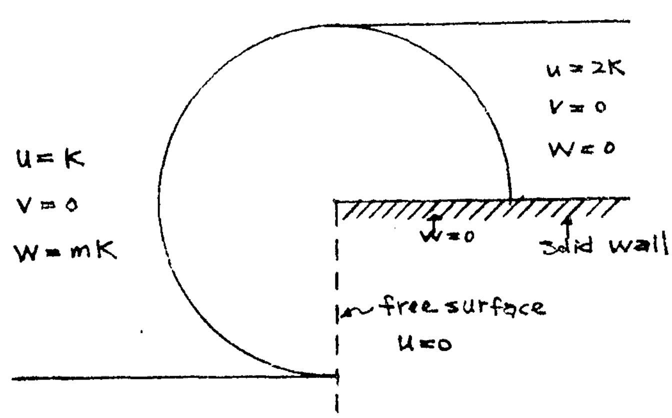
The value of k_p along the solid wall has been plotted in terms of physical coordinates η , ω. (See Fig. 3, p. 49). A set of constant pressure lines (isobar) is also plotted (See Fig. 4, p. 50).

Here, similar to what we have in case A, the line AC behaves like an expansion wave and BC, like a compression wave, or continuation of the reflected shock wave.



(B.12)

Case C:

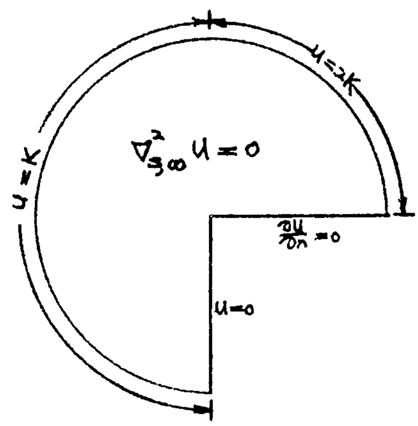


(C.1)

Here to find u, one has to solve

$$\nabla^2_{\zeta\omega} u = 0 \qquad (C.2)$$

with boundary conditions given in Fig. (C.3)

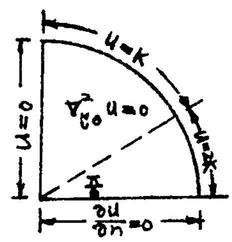


(C.3)

Use conformal transformations

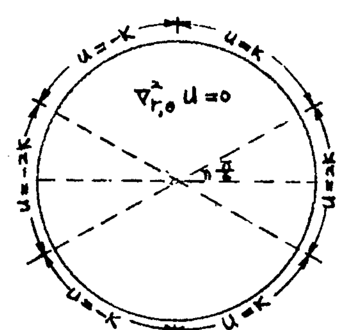
$$\varsigma = \sigma^{\frac{1}{3}} \tag{C.4}$$

where $\varsigma = re^{i\theta}$, $\sigma = se^{i\omega}$. Problem (C.3) is transformed to



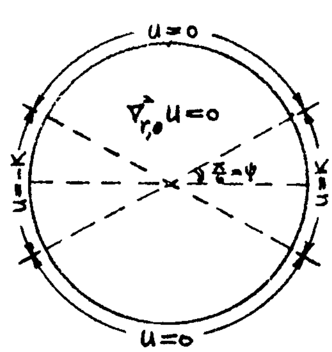
(C.5)

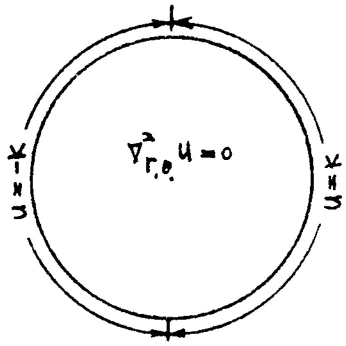
which has same solution as that of



(C.6)

Problem (C.6) may be considered as a superposition of the following two problems: (C.7) and (C.8).





(C.8)

These two problems are same as that of Fig. (A.8) except here $\psi = \frac{\pi}{6}, \frac{\pi}{2}$ for (C.7) and (C.8) respectively and boundary value of u is reduced by one-half. Hence by (A.12), solution of (C.7) will be

$$u = \frac{k}{\pi} \tan^{-1} \left\{ \frac{2r(1-r^2)\cos\theta}{r^4 + r^2(1-4\cos^2\theta)+1} \right\} \equiv \frac{k}{\pi} \vartheta_1 \tag{C.9}$$

where

$$\begin{cases} 0 \le \vartheta_1 < \pi & \text{when } 0 \le \theta \le \frac{\pi}{2}, \quad r < 1 \\ \vartheta_1 = \pi & \text{when } 0 \le \theta \le \frac{\pi}{6}, \quad r = 1 \\ \vartheta_1 = 0 & \text{when } \frac{\pi}{6} \le \theta \le \frac{\pi}{2}, \quad r = 1 \end{cases} \tag{C.10}$$

Solution of (C.8) is

$$u = \frac{2k}{\pi} \tan^{-1} \left(\frac{2r\cos\theta}{1-r^2} \right) \equiv 2\frac{k}{\pi} \vartheta_2 \tag{C.11}$$

Where $0 \le \vartheta_2 \le \frac{\pi}{2}$, when $0 \le \theta \le \frac{\pi}{2}$

Go back to problem (C.3), The solution in (s,ω) is

$$u = -\frac{\pi\alpha}{m} \frac{\vartheta_1 + 2\vartheta_2}{\pi} \tag{C.12}$$

where

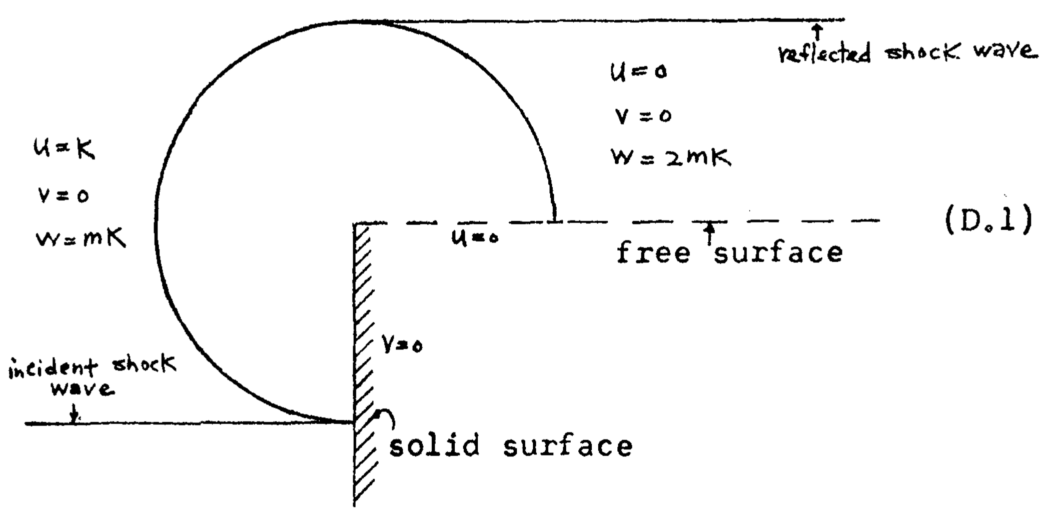
$$\vartheta_1 = \tan^{-1} \left\{ \frac{2s^{\frac{1}{3}}(1-s^{\frac{2}{3}}) \cos\frac{\omega}{3}}{s^{\frac{4}{3}} + s^{\frac{2}{3}}(1-4\cos^2\frac{\omega}{3})+1} \right\}$$

$$
\begin{cases}
0 \le \vartheta_1 < \pi & \text{when } 0 \le \omega \le \frac{3}{2}\pi, \quad S < 1 \\
\vartheta_1 = \pi & \text{when } 0 \le \omega \le \frac{\pi}{2}, \quad S = 1 \\
\vartheta_1 = 0 & \text{when } \frac{\pi}{2} \le \omega \le \frac{3}{2}\pi, \quad S = 1
\end{cases}
$$

and
$$
\vartheta_2 = \tan^{-1}\left\{\frac{2 s^{\frac{1}{3}} \cos\frac{\omega}{3}}{1 - s^{\frac{2}{3}}}\right\}
$$

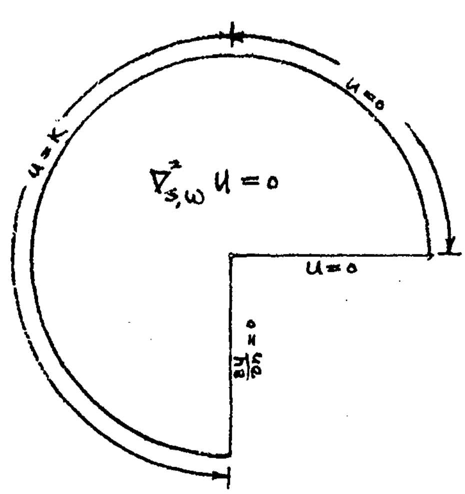
$$
0 \le \vartheta_2 \le \frac{\pi}{2} \qquad \text{when } 0 \le \omega \le \frac{3}{2}\pi, \quad s \le 1
$$

<u>Case D</u>:



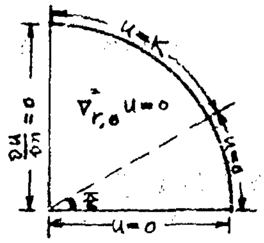
(D.1)

To find u due to edge effect is to solve the following problem:



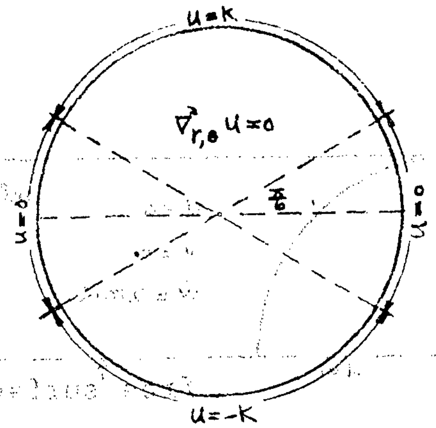
(D.2)

By same conformal transformation formula as in Case C, equ. (C.4), the problem is reduced to



(D.4)

which has same solution as that of



(D.5)

Compare (D.5) with (A.8). If in (A.8), we put $\psi = \frac{\pi}{3}$, replace θ by $(\frac{\pi}{2} + \theta)$ and reduce the boundary value by one-half, we have the problem of (D.5). Thus according to (A.9) and (A.12), we have the solution of (D.5):

$$u = \frac{K}{\pi} \tan^{-1} \left\{ \frac{2\sqrt{3}\, r(1-r^2)\sin\theta}{r^4 - r^2(1+4\sin^2\theta)+1} \right\}$$

$$\equiv \frac{K}{\pi} \Lambda \qquad \text{(D.6)}$$

where

$$\begin{cases} 0 \le \Lambda < \pi & \text{, when } 0 \le \theta \le \frac{\pi}{2},\ r < 1 \\ \Lambda = 0 & \text{, when } 0 \le \theta \le \frac{\pi}{6},\ r = 1 \\ \Lambda = \pi & \text{, when } \frac{\pi}{6} \le \theta \le \frac{\pi}{2},\ r = 1 \end{cases}$$

Transformed back to (s, ω), the solution becomes

$$u = -\frac{\pi\alpha}{m}\,\frac{\Lambda}{\pi} \tag{D.7}$$

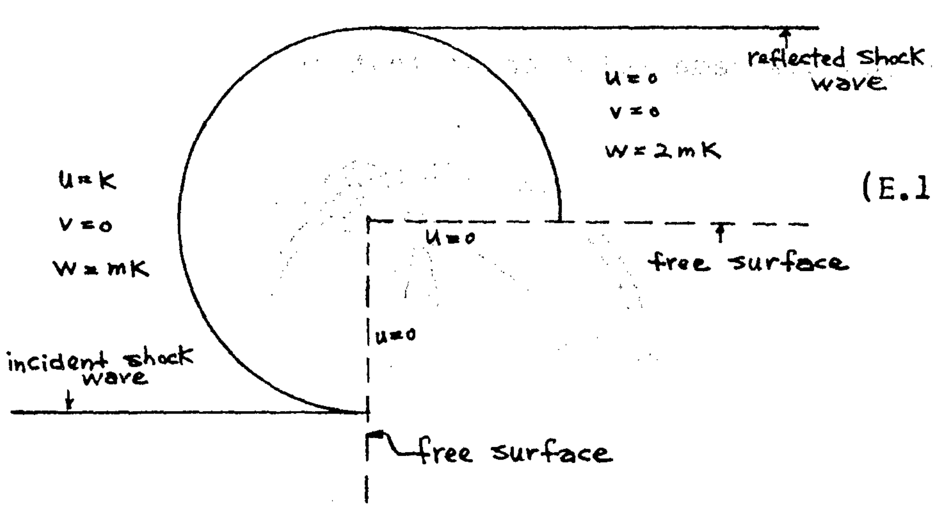
where

$$\Lambda = \tan^{-1}\left\{ \frac{2\sqrt{3}\, s^{\frac{1}{3}}\left(1 - s^{\frac{2}{3}}\right) \sin\frac{\omega}{3}}{s^{\frac{4}{3}} - s^{\frac{2}{3}}\left(1 + 4\sin^2\frac{\omega}{3}\right) + 1} \right\}$$

and

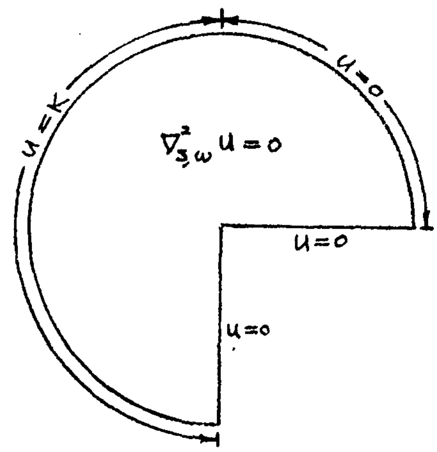
$$\begin{cases} 0 \le \Lambda < \pi & \text{when } 0 \le \omega \le \frac{3}{2}\pi,\ s < 1 \\ \Lambda = 0 & \text{when } 0 \le \omega \le \frac{\pi}{2},\ s = 1 \\ \Lambda = \pi & \text{when } \frac{\pi}{2} \le \omega \le \frac{3}{2}\pi,\ s = 1 \end{cases}$$

Case E:



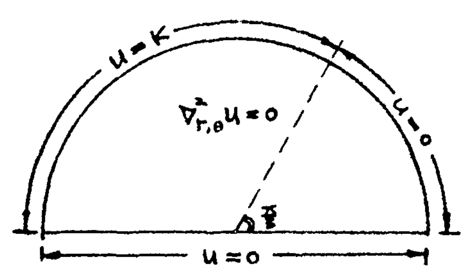
(E.1)

To find u due to edge effect, we have to solve the following problem:



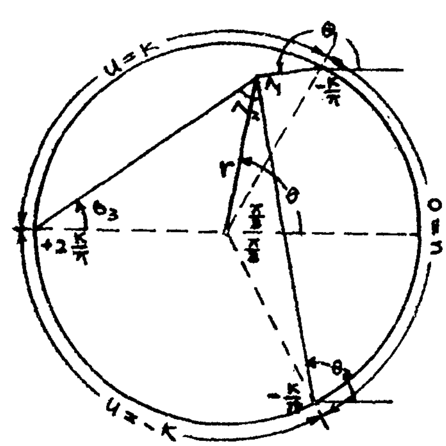
(E.2)

By same transformation formula as used in Case B, the problem is transformed to



(E.3)

which has same solution as that of



(E.4)

Placing vortices as shown in the figure, we find that the solution will be

$$u = \frac{K}{\pi}\left[2\pi - (\Lambda_1 + 2\Lambda_2)\right] \qquad (E.5)$$

where

$$\Lambda_1 = \tan^{-1}\left\{\frac{\sqrt{3}\,(1-2r\cos\theta)}{2r^2 - 2r\cos\theta - 1}\right\}$$

$$\begin{cases} 0<\Lambda_1<\pi \quad \text{when} \quad \frac{\pi}{3} \le \theta \le \pi\,, \quad r \le 1 \\ \frac{\pi}{2} < \Lambda_1 < \frac{3}{2}\pi \quad \text{when} \quad 0 \le \theta \le \frac{\pi}{3}\,, \quad r \le 1 \end{cases}$$

and

$$\Lambda_2 = \tan^{-1}\left\{\frac{\sqrt{3} + \sqrt{3}\,r(\cos\theta + \sqrt{3}\sin\theta)}{(2r^2-1) + r(\cos\theta + \sqrt{3}\sin\theta)}\right\}$$

$$0<\Lambda_2<\pi \quad \text{when} \quad 0 \le \theta \le \pi$$

Transformed back to s,ω, the solution of (E.2) will be

$$u = -\frac{\pi\alpha}{m}\left(2 - \frac{\Lambda_1 + 2\Lambda_2}{\pi}\right) \qquad (E.6)$$

where

$$\Lambda_1 = \tan^{-1}\left\{\frac{\sqrt{3}\,(1-2s^{\frac{1}{3}}\cos\frac{\omega}{3})}{2s^{\frac{2}{3}} - 2s^{\frac{1}{3}}\cos\frac{\omega}{3} - 1}\right\}$$

$$\begin{cases} 0<\Lambda_1<\pi \quad \text{when} \quad \frac{\pi}{2} \le \omega \le \frac{3}{2}\pi \\ \frac{\pi}{2} < \Lambda_1 < \frac{3}{2}\pi \quad \text{when} \quad 0 \le \omega \le \frac{\pi}{2} \end{cases}$$

and

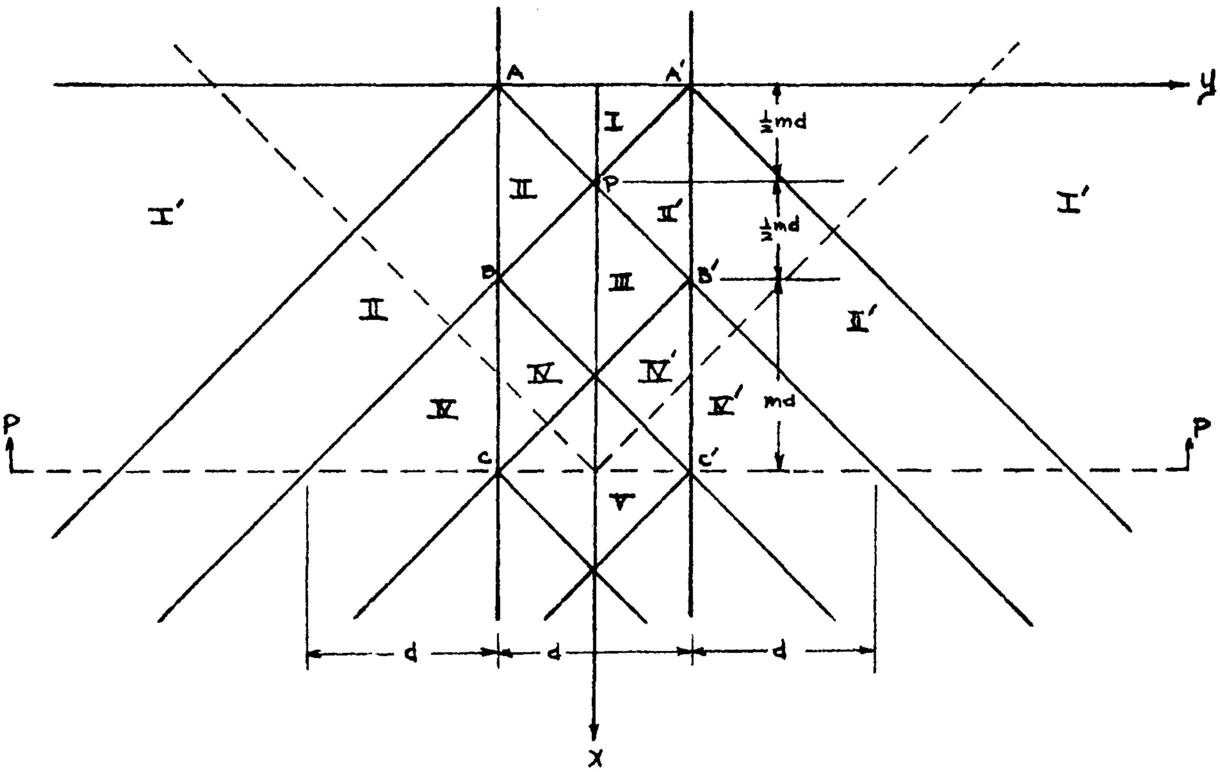
$$\Lambda_2 = \tan^{-1}\left\{\frac{\sqrt{3} + \sqrt{3}\,s^{\frac{1}{3}}(\cos\frac{\omega}{3} + \sqrt{3}\sin\frac{\omega}{3})}{(2s^{\frac{2}{3}}-1) + s^{\frac{1}{3}}(\cos\frac{\omega}{3} + \sqrt{3}\sin\frac{\omega}{3})}\right\}$$

$$0<\Lambda_2<\pi\,, \quad \text{when} \quad 0 \le \omega \le \frac{3}{2}\pi$$

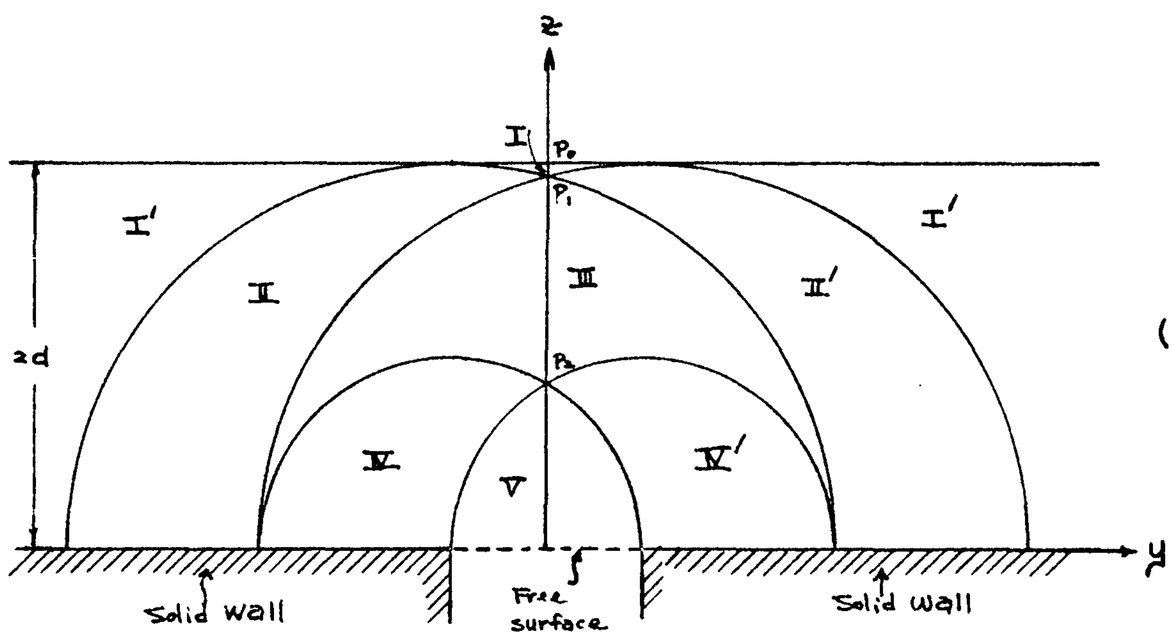
PART III

SHOCK REFLECTED FROM SINGLE SLOT IN INFINITE SOLID WALL

Consider a shock reflected from a slot in an infinite wall, i.e. from a strip of free surface between solid walls, which are infinite in extent, in both sides. As mentioned in the last part, the flow characteristics after reflection at AA' (Fig. 3.1) will be three-dimensional within the Mach cone. It is conical first. Then after a distance $\frac{md}{2}$ (d is the width of slot) from AA', the two waves from A, A' will intersect at p and the flow behind this will start to be a superposition of these two conical waves. The flow is thus no longer conical. More farther down-stream, at a distance md from AA' the waves radiating from A, A' reaches opposite edges B', B respectively. Behind that, if allowed to continue unaltered, the waves would produce a non-vanishing normal velocity on solid wall, which is physically impossible. To remove that, one says two new waves begin to radiate from B', B with such a strength that it will annul the non-vanishing normal velocity on solid wall and leave the zero pressure ($u = o$) condition in the Slot unaltered. These new waves will reach c, c' after another distance md and produces more new waves.



(3.1)



(3.2)

Cross-section along PP in Fig (3.1)

The flow in any one of the regions in Fig. (3.1) will be only influenced by the waves radiating from the points which are within the fore cone of the particular region. For example, the flow in region V will be a superposition of the two-dimensional wave, the initial circular waves from A, A' and the new circular waves from B, B'. Care should be taken in carrying out the superpositions such that the proper boundary conditions are satisfied.

Collecting the results obtained in Part II, we have:

in I:

$$u = 0$$
$$v = 0 \qquad (3.3)$$
$$w = -2\varpi\alpha$$

in I':

$$u = -2\frac{\varpi\alpha}{m}$$
$$v = 0 \qquad (3.4)$$
$$w = 0$$

in II:

$$u_{I} = -\frac{2\varpi\alpha}{m}\,\frac{\Lambda}{\pi} \qquad (3.5)$$

where

$$\Lambda = \tan^{-1}\left\{\frac{2\sqrt{m\varpi(x - m\varpi)}\,\sin\frac{\omega}{2}}{x - 2m\varpi\sin^2\frac{\omega}{2}}\right\}$$

$$
\begin{cases}
0 \le \Lambda < \pi & \text{when } 0 \le \omega \le \pi \,, \; m\varpi < x \\
\Lambda = 0 & \text{when } 0 \le \omega \le \frac{\pi}{2} \,, \; m\varpi = x \\
\Lambda = \pi & \text{when } \frac{\pi}{2} \le \omega \le \pi \,, \; m\varpi = x
\end{cases}
$$

$$
V_{\mathrm{II}} = -\frac{4\pi\alpha}{\pi}\sqrt{\frac{x-m\varpi}{m\varpi}}\,\sin\frac{\omega}{2} \tag{3.6}
$$

$$
W_{\mathrm{II}} = -\frac{4\pi\alpha}{\pi}\sqrt{\frac{x-m\varpi}{m\varpi}}\,\cos\frac{\omega}{2} - \frac{2\pi\alpha}{\pi}\vartheta
$$

where

$$
\vartheta = \tan^{-1}\left\{\frac{2\sqrt{m\varpi(1-m\varpi)}\,\cos\frac{\omega}{2}}{x - 2m\varpi\cos^2\frac{\omega}{2}}\right\} \tag{3.7}
$$

$$
\begin{cases}
0 \le \vartheta < \pi & \text{when } 0 \le \omega \le \pi \,, \; m\varpi < x \\
\vartheta = 0 & \text{when } \frac{\pi}{2} \le \omega \le \pi \,, \; m\varpi = x \\
\vartheta = \pi & \text{when } 0 \le \omega \le \frac{\pi}{2} \,, \; m\varpi = x
\end{cases}
$$

In above

$$
\varpi = \sqrt{\left(y+\frac{d}{2}\right)^2 + z^2} \tag{3.8}
$$

$$
\omega = \tan^{-1}\left(\frac{z}{y+\frac{d}{2}}\right) \tag{3.9}
$$

here $0 < \omega < \pi$

$\omega = 0$ when $z = 0, \; y > -\frac{d}{2}$

$\omega = \pi$ when $z = 0, \; y < -\frac{d}{2}$

in II':

$$
U_{\mathrm{II}'} = -\frac{2\pi\alpha}{m}\,\frac{\Lambda}{\pi}
$$

where [illegible] (3.10)

$$\begin{cases} 0 \le \Lambda < \pi \text{ when } 0 \le \omega \le \pi, & m\varpi < x \\ \Lambda = \pi \text{ when } 0 \le \omega \le \frac{\pi}{2}, & m\varpi = x \\ \Lambda = 0 \text{ when } \frac{\pi}{2} \le \omega \le \pi, & m\varpi = x \end{cases}$$

$$V_{II'} = -\frac{4\pi\alpha}{\pi}\sqrt{\frac{x-m\varpi}{m\varpi}}\cos\frac{\omega}{2} \tag{3.11}$$

$$W_{II'} = -\frac{4\pi\alpha}{\pi}\sqrt{\frac{x-m\varpi}{m\varpi}}\sin\frac{\omega}{2} - \frac{2\pi\alpha}{\pi}\vartheta \tag{3.12}$$

$$\text{where } \vartheta = \tan^{-1}\left\{\frac{2\sqrt{m\varpi(1-m\varpi)}\,\sin\frac{\omega}{2}}{x - 2m\varpi\sin^2\frac{\omega}{2}}\right\}$$

$$\begin{cases} 0 \le \vartheta < \pi, & \text{when } 0 \le \omega \le \pi, \quad m\varpi < x \\ \vartheta = \pi, & \text{when } \frac{\pi}{2} \le \omega \le \pi, \quad m\varpi = x \\ \vartheta = 0, & \text{when } 0 \le \omega \le \frac{\pi}{2}, \quad m\varpi = x \end{cases}$$

In above

$$\varpi = \sqrt{\left(y - \frac{d}{2}\right)^2 + z^2} \tag{3.13}$$

$$\omega = \tan^{-1}\left(\frac{z}{y-\frac{d}{2}}\right) \tag{3.14}$$

$$0 < \omega < \pi$$

$$\omega = 0 \text{ when } z = 0, \quad y > \frac{d}{2}$$

$$\omega = \pi \text{ when } z = 0, \quad y < \frac{d}{2}$$

in III:

Symbolically the solution in region III can be written

III = II + II' - I

The correctness of the result may be verified very easily by consideration of the boundary conditions.

For a point (x, y, z)

$$u_{III} = u_{II} + u_{II'} \tag{3.15}$$

$$v_{III} = v_{II} + v_{II'} \tag{3.16}$$

$$w_{III} = w_{II} + w_{II'} + \pi\alpha \tag{3.17}$$

in IV, IV' and regions further downstream:

To determine the flow of the new wave reflected from the opposite edge, a more complicated problem results, which will be a subject for further study. Some discussions on this problem is given in Part V.

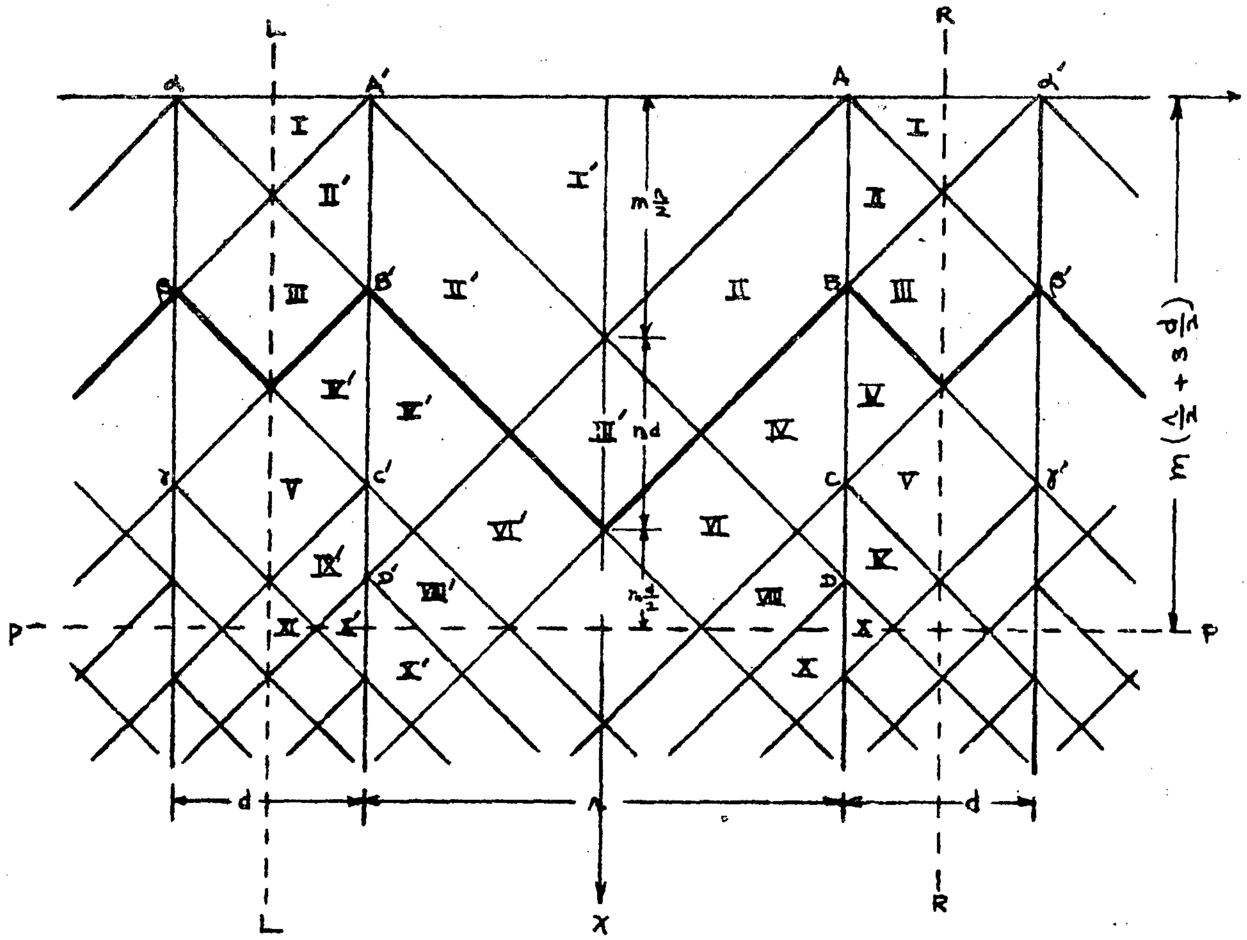
The pressure coefficient c_p along z axis in Fig. (3.2) is plotted for regions I and III at x = 2 md. See Fig. 5, p. 51 and Appendix III.

PART IV

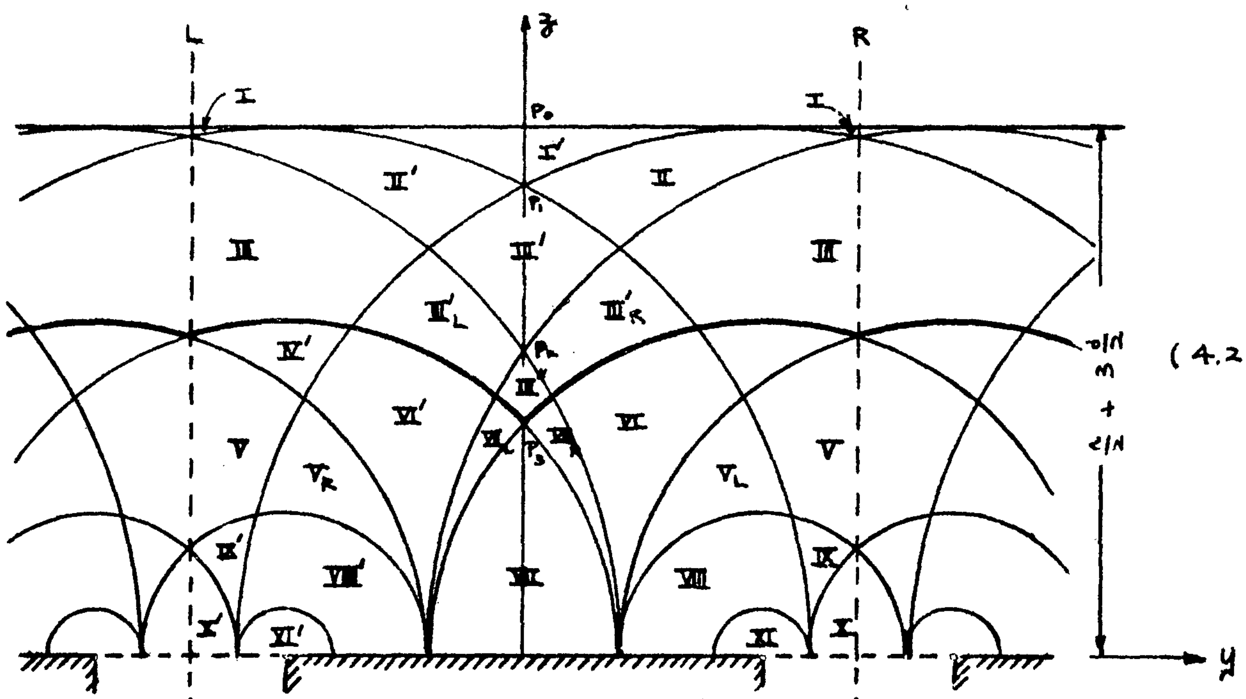
SHOCK REFLECTED FROM MULTIPLY-SLOTTED SOLID WALL

To determine the flow due to a shock reflected from a multiply-slotted surface, we should have, in addition to what we did for free surface slot in Part III, a similar consideration of superpositions and reflections of waves in the solid wall part. Also more new waves will come from other slots. The entire picture of the flow field depends upon the ratio of the width of slot, d, to that of solid surface between slots, λ; and, moreover, on M_∞. For practical interest, that ratio $\frac{d}{\lambda}$ is relatively small. In Fig. 4.1, 4.2, $\frac{d}{\lambda} = \frac{1}{2.5}$, $M_\infty = \sqrt{2}$ are used.

As shown in Fig. 4.1, 4.2, waves radiating from A, A' behaves at that in Part III first. After a distance $m\lambda$ from A A', the wave from A (or A') reaches edges A' D' (or A D), and if allowed to continue unaltered, would produce a non-vanishing u on the free surface. To annul that u, a new wave starts to radiate from D' (or D) which will satisfy the condition w = o on solid wall at the same time. A more complicate flow pattern will naturally result from super positions of the new waves and the additional waves from neighboring slots. The



(4.1)



(4.2)

cross-section along PP in Fig 4.1

situation will become even much more complicated when we go further down-stream. Assume the slots and the solid walls are uniformly spaced with constant d/λ ratio. Then, as shown clearly in the figure, the flow above the slot will be symmetrical about the vertical perpendicular plane bisecting the slot and the flow above the solid wall, symmetrical about the vertical perpendicular plane bisecting the solid wall spacing. Moreover, we need only to specify the flow characteristic between the LL and RR lines in Fig. 4.1. The flow right or left to that region is just a repetition of this typical one.

Same as in Fig. (3.1), the flow in any region in Fig. (4.1) will be a superposition of waves radiating from the points which are within the fore cone of that particular region. By means of results of Case A, Part II, the flow characteristics of the regions which are not influenced by waves from B, B', β, β' or further downstream points can be determined. That is, we know all these regions which are above or forward of the extra-heavy lines in Fig. 4.1, 4.2.

Use the coordinate system as indicated in Fig. 4.1, 4.2. Formulas of u, v, w in regions I, I', II, II', III are same as that in Part III, i.e. (3.3) to (3.17) inclusively, except that in (3.8), (3.9), (3.13), (3.14), d should be replaced by -r.

In III', for a given point (x, y, z),

$$u_{III'} = u_{II} + u_{II'} + \frac{2\varpi\alpha}{m}$$

$$v_{III'} = v_{II} + v_{II'}$$

$$w_{III'} = w_{II} + w_{II'}$$

In III'_L, III'_R and III", similar formula may be written except that appropriate ϖ, ω should be used.

The pressure coefficient c_p along z axis in Fig. (4.2) has been plotted for regions I', III', III" at $x = m/2\ (\Omega + 3d)$. See Fig. 6, p.52. Detail calculation is given in Appendix IV. (P.61)

PART V

DISCUSSION

For the problem of a shock reflected from a single slot, there results, as has already been pointed out in Part II, pairs of new waves when the original waves radiating from one side of slot reach the opposite side. This is also true for the problem of a flat plate wing of very low aspect ratio (less than 1 for $M_\infty = \sqrt{2}$) at an angle of attack α. The pressure distribution of the latter problem has been calculated by Gunn (reference 4) up to 2.5d (d is the span of wing) from the leading edge, and by Stewartson (reference 5) with an asymptotic expression. There arises the question:* Are these two problems similar, and if so, can we borrow some results of Gunn'sor Stewartson's solution for our problem?

The differential equation governing the velocity potential or the velocity components u, v, w of either problem is a linearized one. Any solution which satisfie the differential equation may be superposed to form a ne solution, provided the appropriate boundary conditions are satisfied. This is the so called "cancellation win method. By this method, for example, the lift distribu

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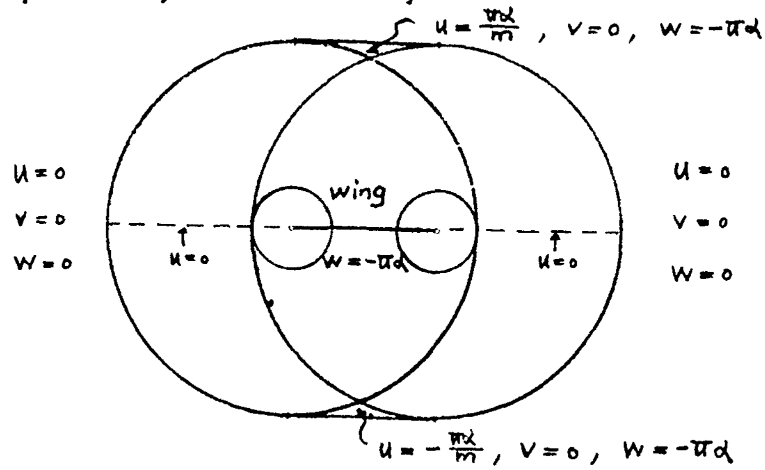
* Question suggested by Prof. W. R. Sears.

on a given wing may be determined by cancelling excess lift on a related wing with a known loading; i.e. the problem can be expressed as the two-dimensional wing problem plus a cancellation wing. For the narrow flat plate problem, the boundary conditions are:



(5.1)

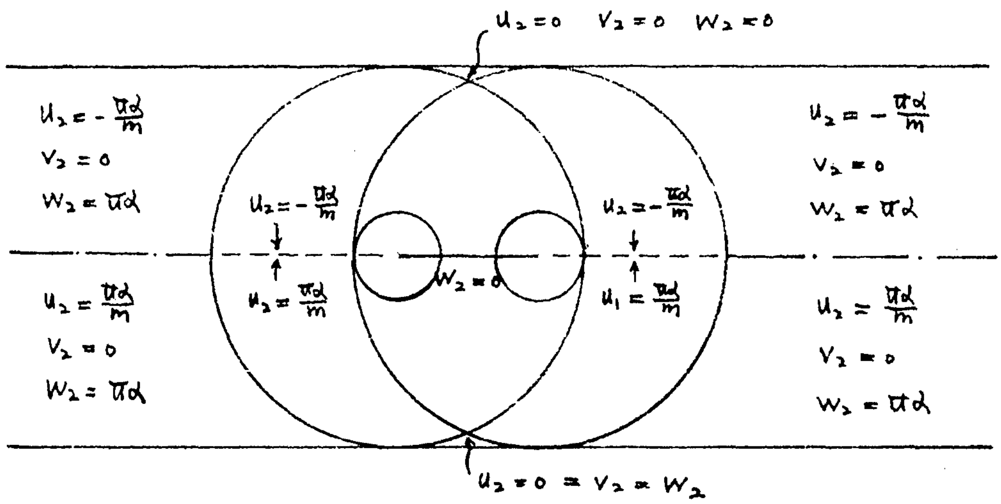
which may be considered as sum of the 2-dimensional flow:

$$u_1 = \frac{\pi\alpha}{m}$$
$$v_1 = 0$$
$$w_1 = -\pi\alpha$$

$\infty \leftarrow$ ——————————————— $\rightarrow \infty$ (5.2)

$$u_1 = -\frac{\pi\alpha}{m}$$
$$v_1 = 0$$
$$w_1 = -\pi\alpha$$

plus the "cancellation wing" problem:



(5.3)

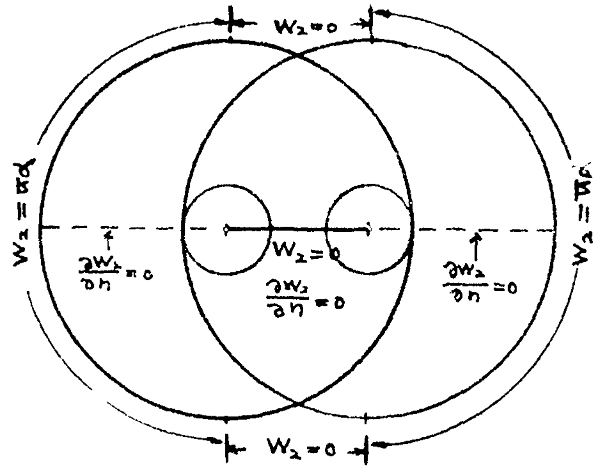
The W component of the problem is symmetrical about the plane of wing (the x-y plane for small-perturbation approximation). Thus we have

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial n} = 0 \qquad \text{throughout the x-y plane.}$$

The problem of finding W_2 of (5.3) is to solve

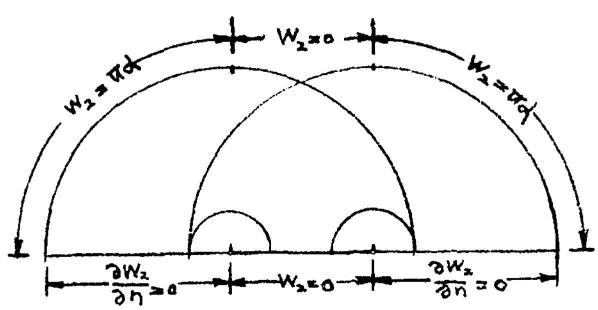
$$m^2 \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial z^2} = 0 \tag{5.4}$$

with the following boundary conditions: (Fig 5.5, next page)



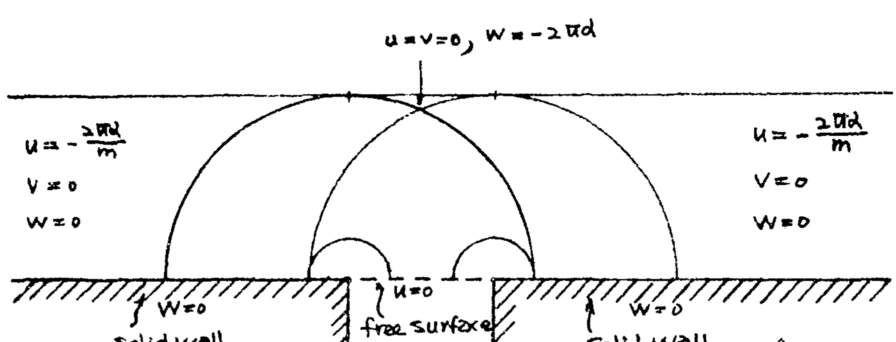
(5.5)

or just to solve (5.4) for the upper half region of (5.5), due to $\frac{\partial W_2}{\partial n} = 0$, i.e.



(5.6)

Now, for the slot problem, we have

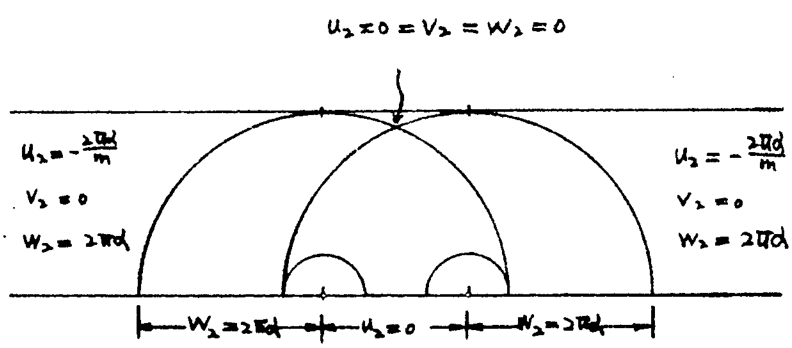


(5.7)

which may be also considered as the sum of 2-dimensional flow.

$$u_1 = 0 \qquad v_1 = 0 \qquad w_1 = -2\pi\alpha \tag{5.8}$$

and the "cancellation wing" problem:

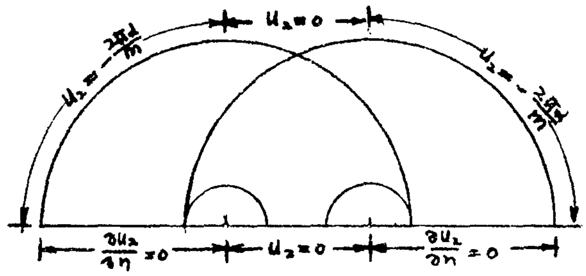


(5.9)

By irrotationality condition, the problem to find u_2 of (5.9) is to solve

$$m^2 \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial y^2} - \frac{\partial^2 u_2}{\partial z^2} = 0 \tag{5.10}$$

with the following boundary conditions:



(5.11)

compare (5.4) with (5.10), also (5.6) with (5.11). The differential equations are identical. The boundary conditions would be also identical if $m = 1$ (i.e. $M_\infty = \sqrt{2}$) and -2α in (5.11) is replaced by α. In other words, it seems that the expression of W component of the narrow flat plate airfoil may be used as that of u component of our slot problem; except some changes of constants. However, as we have shown it clearly in Case A, Part I, there may be some singularity in W while the assumption of finite pressure rules out the possibility of singularity in u. The possible existence of singularity makes the solution of problem not unique. Hence the u-W correspondence breaks down finally.

The above consideration may be best illustrated by the semi-infinite rectangular wing (infinite toward left) at an angle of attack α with comparison to Case A in Part II. The W component of such a wing has been computed

by Gunn (reference 4, p. 338, note that results given there are for wing extended to infinity toward right) by means of Laplace transform. After changing to our notation, W inside Mach cone reads as

$$W = -U\alpha + \frac{U\alpha}{\pi}\left\{\sin^{-1}\sqrt{\frac{m(\varpi+z)}{x+mz}} + \sin^{-1}\sqrt{\frac{m(\varpi-z)}{x-mz}}\right\} + \frac{2U\alpha}{\pi}\sqrt{\frac{x}{m\varpi}-1}\ \cos\frac{\omega}{2}$$

$$= -U\alpha + \frac{U\alpha}{\pi}\tan^{-1}\left\{\frac{\sqrt{2m\eta(1-m\eta)}\ \cos\frac{\omega}{2}}{1-2m\eta\cos^2\frac{\omega}{2}}\right\} + \frac{2U\alpha}{\pi}\sqrt{\frac{1-m\eta}{m\eta}}\cos\frac{\omega}{2} \qquad (5.12)$$

By consideration given above, the W_2 of cancellation wing is, putting m = 1,

$$W_2 = \frac{U\alpha}{\pi}\tan^{-1}\left\{\frac{\sqrt{2\eta(1-\eta)}\ \cos\frac{\omega}{2}}{1-2\eta\cos^2\frac{\omega}{2}}\right\} + \frac{2U\alpha}{\pi}\sqrt{\frac{1-\eta}{\eta}}\cos\frac{\omega}{2} \qquad (5.13)$$

In (5.13), change α into -2α, we have u_2 of "cancellation wing" of slot problem for m=1. Since u_1 = o, the u component of slot problem at M_∞ will be equal to u_2, i.e.

$$u = -\frac{2U\alpha}{m\pi}\tan^{-1}\left\{\frac{\sqrt{2m\eta(1-m\eta)}\ \cos\frac{\omega}{2}}{1-2m\eta\cos^2\frac{\omega}{2}}\right\} - \frac{4U\alpha}{m\pi}\sqrt{\frac{1-m\eta}{m\eta}}\cos\frac{\omega}{2} \qquad (5.14)$$

The first term on right side of (5.14) is exactly

same as (A.15), p. 12. However, in addition, we have a singularity term here.

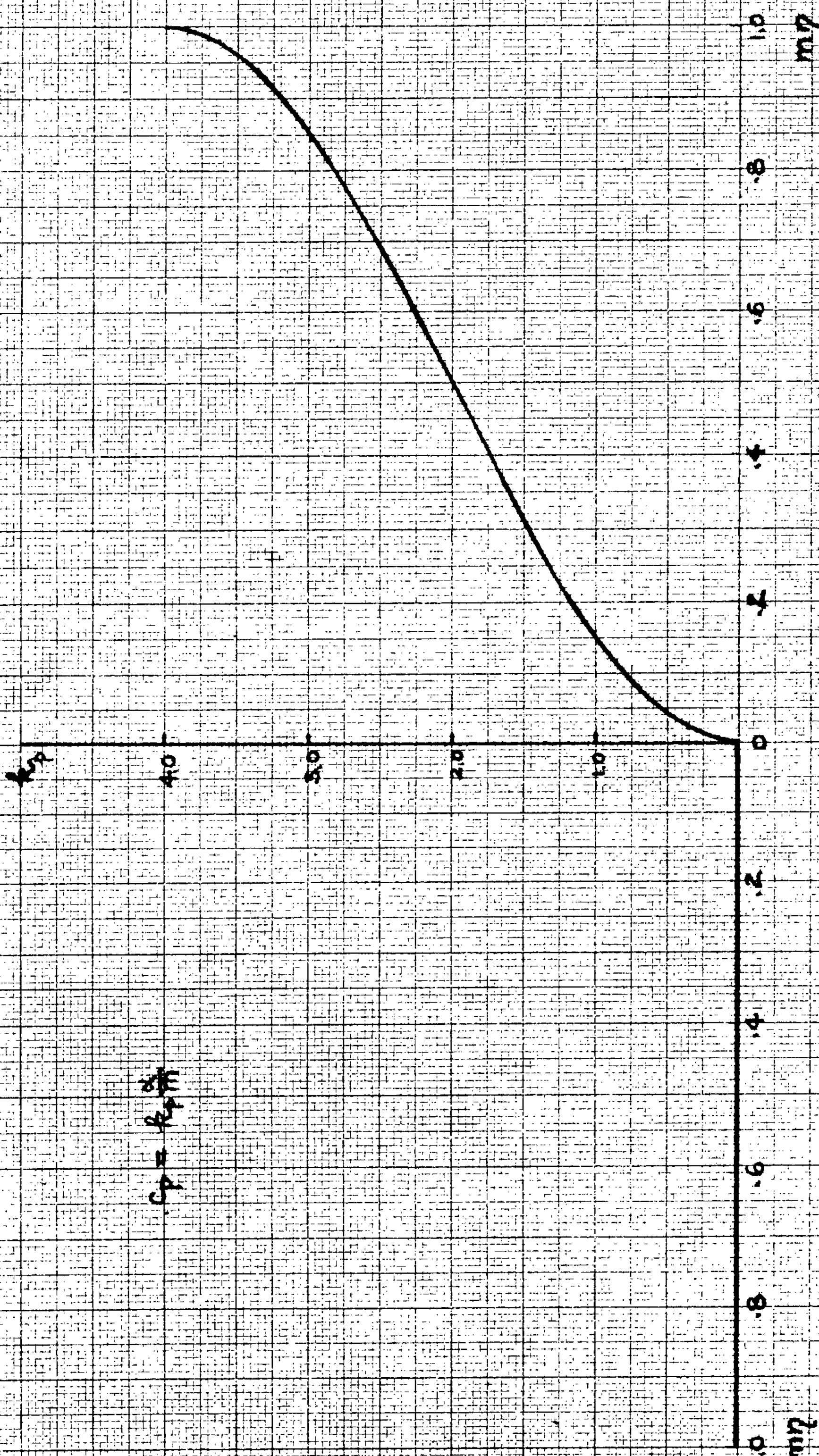


Fig. I

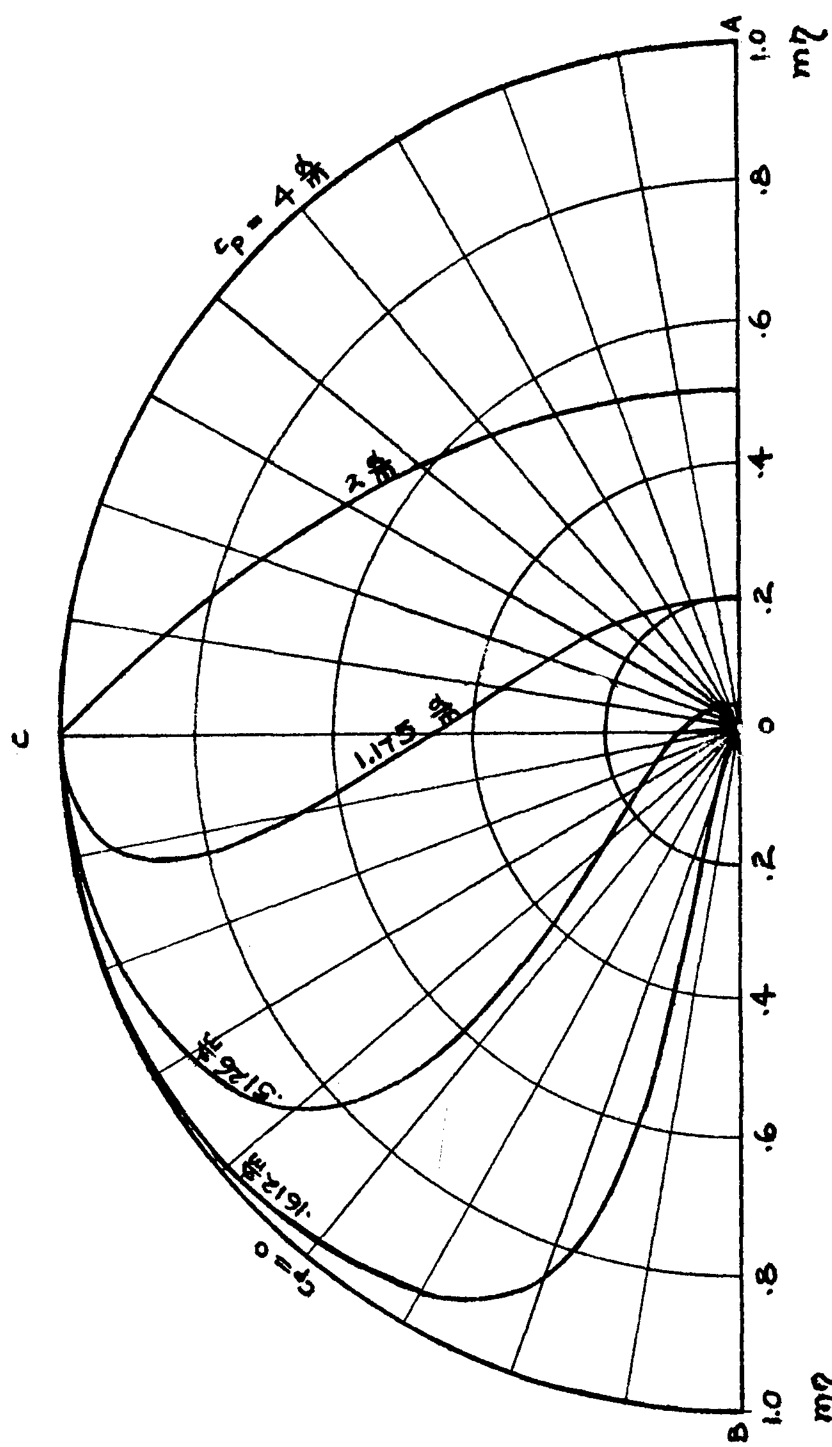
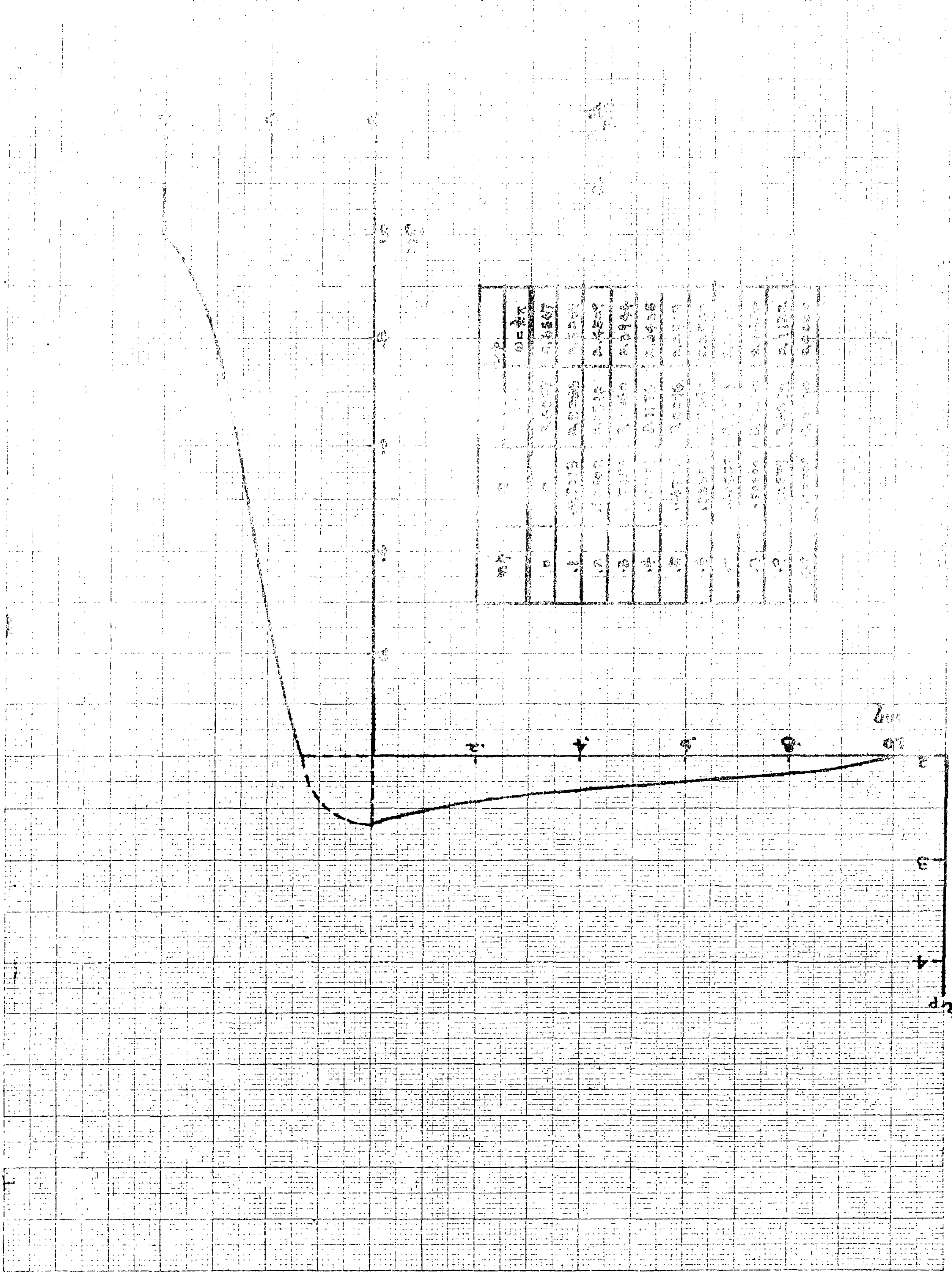


Fig. 2



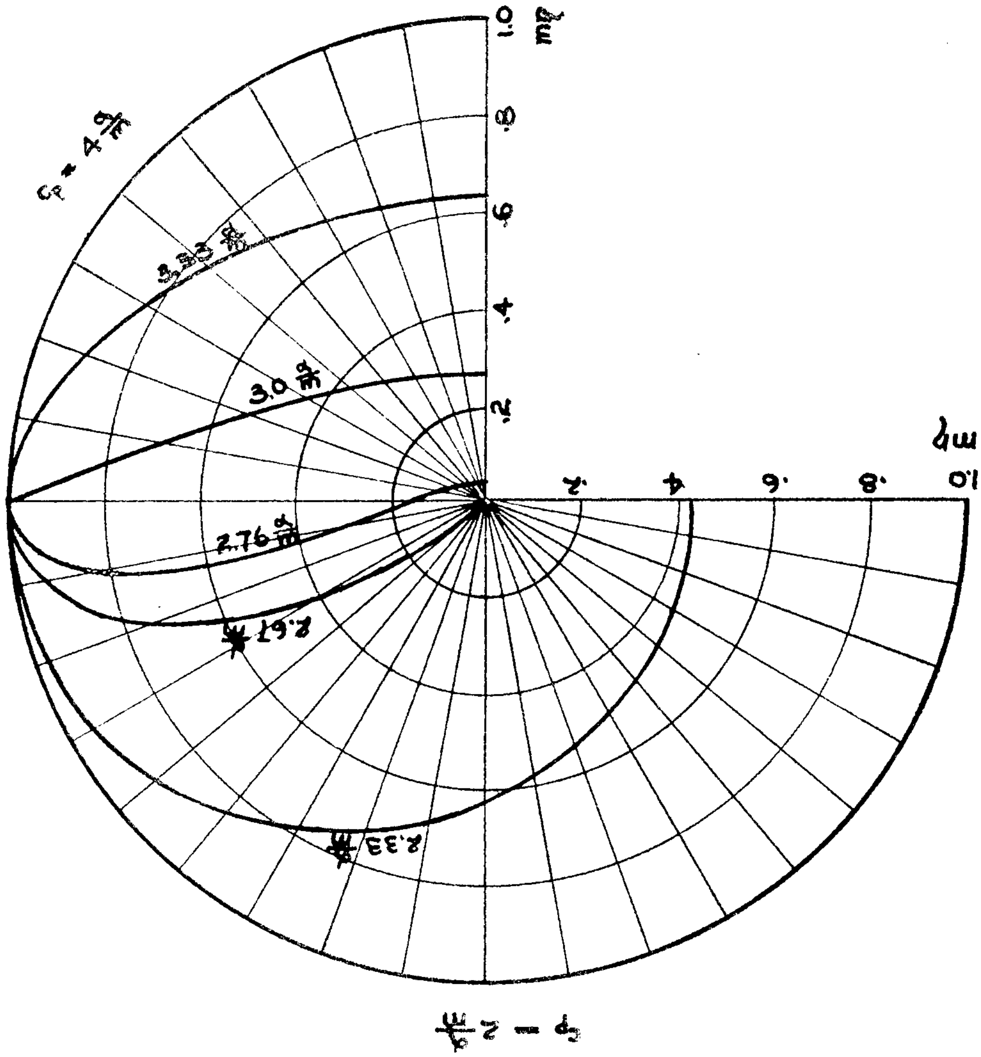


Fig. 4

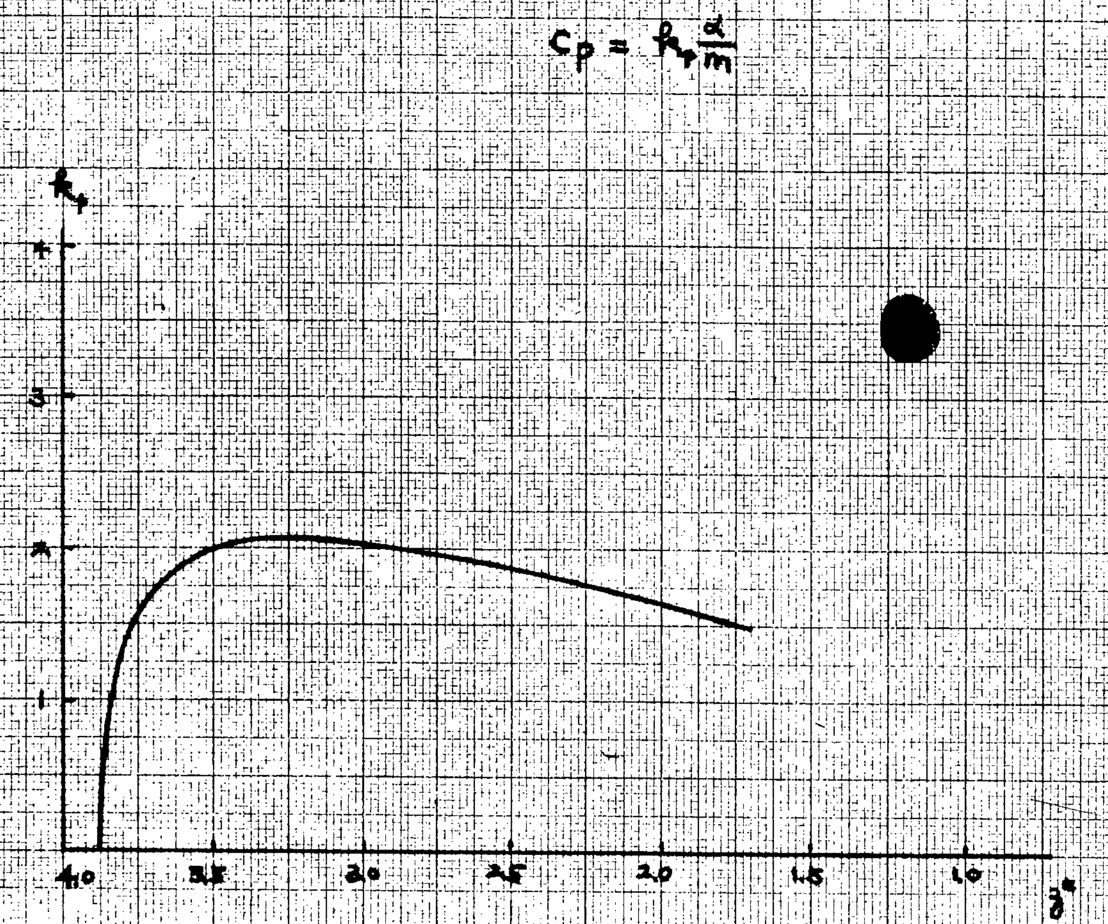


Fig. 5

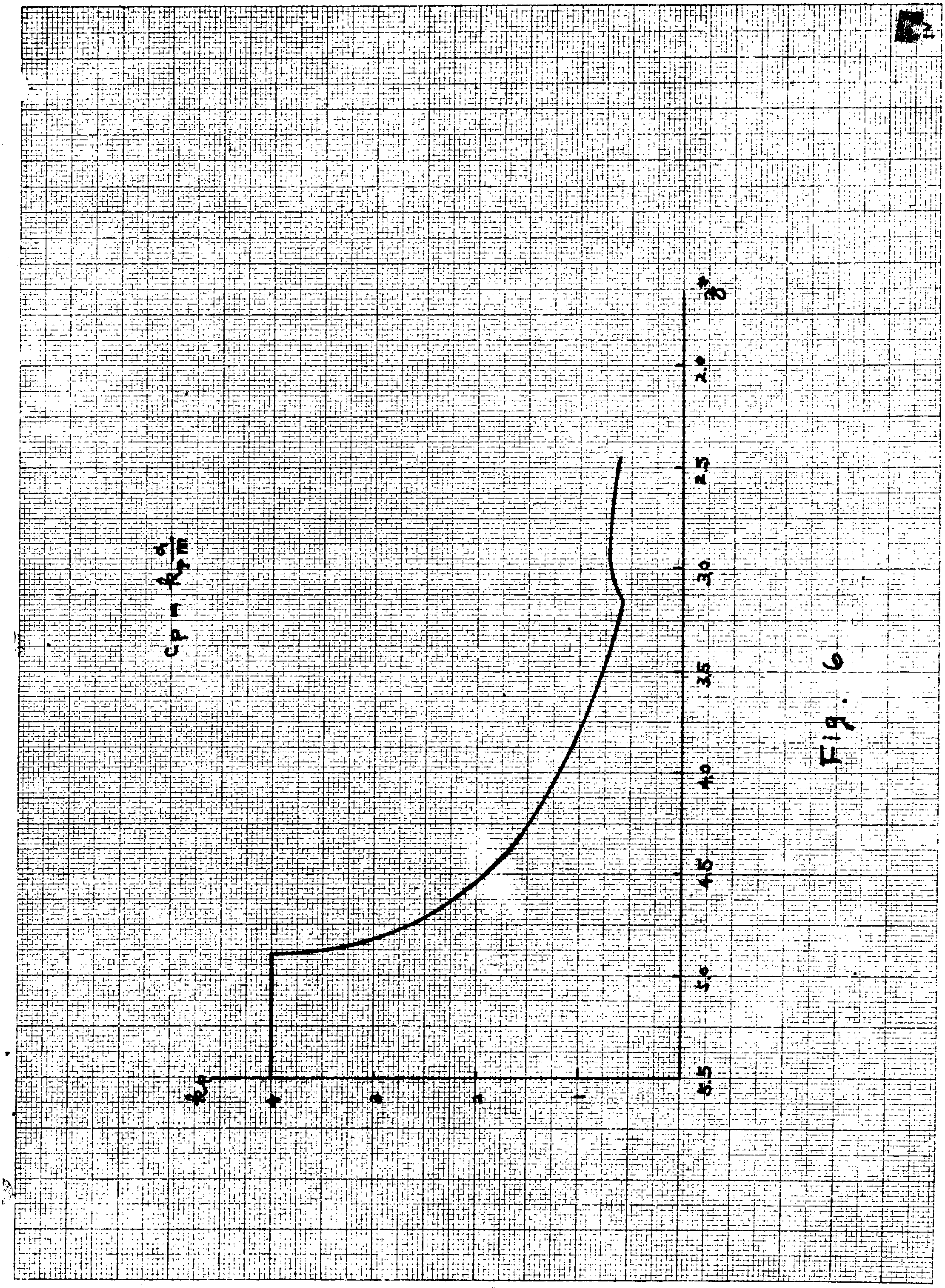


Fig. 6

APPENDIX I

To plot the constant pressure lines or iso-bars,(Fig. 2) rewrite (A.15, P 12)

$$c_p = \frac{4\alpha}{m\pi} \tan^{-1}\left\{ \frac{2\sqrt{m\eta(1-m\eta)}\,\cos\frac{\omega}{2}}{1-2m\eta\cos^2\frac{\omega}{2}} \right\} \qquad (1)$$

in the form

$$m\eta(1-m\eta)\cos^2\frac{\omega}{2} = p^2\left(1-2m\eta\cos^2\frac{\omega}{2}\right)^2 \qquad (2)$$

where $$p \equiv \frac{1}{2}\tan\left(\frac{c_p}{4\frac{\alpha}{m}}\pi\right) \qquad (3)$$

By some algebraic manipulation, (2) may be written as

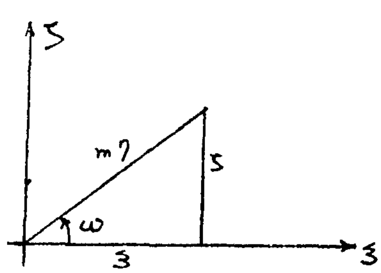
$$m\eta = \frac{1+4p^2 \pm \sqrt{1+\frac{8p^2\cos\omega}{1+\cos\omega}}}{2\left[1+2p^2(1+\cos\omega)\right]} \qquad (4)$$

Give different value to c_p, i.e. to p, we may plot a set of curves of $m\eta$ vs. ω. That results the Fig. 2. p. 48.

To find the slope of the iso-bar, introduce ξ, ζ, [illegible], such that

$$\sqrt{\xi^2+\zeta^2} = m\eta \qquad (5)$$

$$\tan^{-1}\frac{\zeta}{\xi} = \omega \qquad (6)$$



Denote $P = 2p^2$ and by the formula

$$2\cos^2\frac{\omega}{2} = 1 + \cos\omega = 1 + \frac{\xi}{m\eta} \tag{7}$$

We may rewrite (2) as

$$\left[1-\sqrt{\xi^2+\zeta^2}\right]\left[\xi+\sqrt{\xi^2+\zeta^2}\right] = P\left(1-\xi-\sqrt{\xi^2+\zeta^2}\right)^2 \tag{8}$$

or

$$(1-\xi)(1+2P)\sqrt{\xi^2+\zeta^2} = \zeta^2(1+P) + \left[(\xi^2-\xi)(1+2P)+P\right] \tag{9}$$

squaring and collecting, we have

$$\zeta^4(1+P)^2 + \zeta^2(1+P)\left(\xi^2+2P\xi - \frac{1+2P+2P^2}{1+2P}\right)$$
$$+ \left\{P^2 - 2P(1+2P)\xi(1-\xi)\right\} = 0 \tag{10}$$

Hence

$$\frac{d\zeta}{d\xi} = \frac{-2(1+2P)(P+\xi)\zeta^2 + 2P(1+2P)(1-2\xi)}{4(1+P)^2\zeta^3 + 2\zeta(1+2P)\left(\xi^2+2P\xi-\frac{1+2P+2P^2}{1+2P}\right)} \tag{11}$$

At C: $\zeta = 1$, $\xi = 0$

$$\frac{d\zeta}{d\xi} = \frac{0}{2(1+2P)} \tag{12}$$

Thus the slopes of all iso-bars are zero at C, except possibly not for iso-bars of $P = 0$ or ∞.

For $P = 0$, by (8), we have

$$1 - \sqrt{\xi^2+\zeta^2} = 0 \tag{13}$$

or

$$\xi + \sqrt{\xi^2+\zeta^2} = 0 \tag{14}$$

Both (13) and (14) give $\frac{d\zeta}{d\xi} = 0$ at C.

For $P = \infty$, i.e. $c_p = 2\,\alpha/m$, we have, by (8)

$$1 - \zeta - \sqrt{\zeta^2 + s^2} = 0 \qquad (15)$$

or

$$s = \sqrt{1 - 2\zeta} \qquad (16)$$

Thus at c:

$$\left(\frac{ds}{d\zeta}\right)_{at\,c} = \left(-\frac{1}{\sqrt{1-2\zeta}}\right)_{\substack{s=1\\ \zeta=0}} = -1 \qquad (17)$$

Hence the tangent of the iso-bar of $c_p = 2\alpha/m$ at C makes an angle of 135^0 with the horizontal axis which is parallel to the boundary surface.

APPENDIX II

To evaluate the integration of

$$W = \frac{2\pi\alpha}{\pi}\sin\frac{\omega}{2}\int \frac{1-m\eta + 2m^2\eta^2\cos\omega\cos^2\frac{\omega}{2}}{(m\eta)^{\frac{3}{2}}(1-m\eta)^{\frac{1}{2}}(1-m^2\eta^2\sin^2\omega)}\,d(m\eta)$$

write

$$m\eta \equiv X \equiv \sin^2\theta$$

Then

$$W = \frac{4\pi\alpha}{\pi}\sin\frac{\omega}{2}\int \frac{\cos^2\theta + \sin^4\theta\,(2\cos\omega\cos^2\frac{\omega}{2})}{\sin^2\theta\,(1-\sin^4\theta\sin^2\omega)}\,d\theta$$

$$= \frac{4\pi\alpha}{\pi}\sin\frac{\omega}{2}\int\left[\frac{1}{\sin^2\theta\,(1-\sin^4\theta\sin^2\omega)} - \frac{1}{(1-\sin^4\theta\sin^2\omega)}\right.$$

$$\left. + \frac{\sin^2\theta\,(2\cos\omega\cos^2\frac{\omega}{2})}{(1-\sin^4\theta\sin^2\omega)}\right]d\theta$$

$$= \frac{2\pi\alpha}{\pi}\sin\frac{\omega}{2}\left\{2\int\frac{d\theta}{\sin^2\theta} - \frac{1+\cos\omega+\sin\omega}{\sin\omega}\int\frac{d\theta}{1+\sin^2\theta\sin\omega}\right.$$

$$\left. + \frac{1+\cos\omega-\sin\omega}{\sin\omega}\int\frac{d\theta}{1-\sin^2\theta\sin\omega}\right\}$$

As

$$\int\frac{d\theta}{\sin^2\theta} = -\cot\theta$$

$$\int\frac{d\theta}{1\pm\sin\omega\sin^2\theta} = \frac{1}{\sqrt{1\pm\sin\omega}}\tan^{-1}\left(\sqrt{1\pm\sin\omega}\,\tan\theta\right)$$

$$W = -\frac{4\pi d}{\pi}\sin\frac{\omega}{2}\cot\theta - \frac{2\pi d}{\pi}\sin\frac{\omega}{2}\cdot\left(\frac{1+\cos\omega+\sin\omega}{\sin\omega\sqrt{1+\sin\omega}}\right) \times$$

$$\left\{\tan^{-1}\left(\sqrt{1+\sin\omega}\,\tan\theta\right) - \frac{1+\cos\omega-\sin\omega}{1+\cos\omega+\sin\omega}\sqrt{\frac{1+\sin\omega}{1-\sin\omega}}\,\tan^{-1}\left(\sqrt{1-\sin\omega}\,\tan\theta\right)\right\}$$

Owing to

$$\sin\frac{\omega}{2}\cdot\frac{1+\cos\omega+\sin\omega}{\sin\omega\sqrt{1+\sin\omega}} = \sin\frac{\omega}{2}\cdot\frac{2\cos^2\frac{\omega}{2}+2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}\sqrt{1+\sin\omega}} = \frac{\cos\frac{\omega}{2}+\sin\frac{\omega}{2}}{\sqrt{1+\sin\omega}}$$

$$= \sqrt{\frac{(\cos\frac{\omega}{2}+\sin\frac{\omega}{2})^2}{1+\sin\omega}} = 1$$

$$\frac{1+\cos\omega-\sin\omega}{1+\cos\omega+\sin\omega}\sqrt{\frac{1+\sin\omega}{1-\sin\omega}} = \frac{\cos\frac{\omega}{2}-\sin\frac{\omega}{2}}{\cos\frac{\omega}{2}+\sin\frac{\omega}{2}}\cdot\sqrt{\frac{1+\sin\omega}{1-\sin\omega}} = \frac{\cos^2\frac{\omega}{2}-\sin^2\frac{\omega}{2}}{(\cos\frac{\omega}{2}+\sin\frac{\omega}{2})}\cdot\frac{1+\sin\omega}{\sqrt{1-\sin^2\omega}}$$

$$= 1$$

We have

$$W = -\frac{4\pi\alpha}{\pi}\sin\frac{\omega}{2}\cot\theta - \frac{2\pi\alpha}{\pi}\left\{\tan^{-1}\left(\sqrt{1+\sin\omega}\tan\theta\right) - \tan^{-1}\left(\sqrt{1-\sin\omega}\tan\theta\right)\right\}$$

But

$$\tan^{-1}\left(\sqrt{1+\sin\omega}\tan\theta\right) - \tan^{-1}\left(\sqrt{1-\sin\omega}\tan\theta\right) = \tan^{-1}\left\{\frac{\left(\sqrt{1+\sin\omega} - \sqrt{1-\sin\omega}\right)\tan\theta}{1 - \sqrt{1-\sin^2\omega}\tan^2\theta}\right\}$$

and

$$\begin{aligned}
&\sqrt{1+\sin\omega} - \sqrt{1-\sin\omega} \\
&\quad = \sqrt{\left(\sqrt{1+\sin\omega} - \sqrt{1-\sin\omega}\right)^2} = \sqrt{2 - 2\sqrt{1-\sin^2\omega}} \\
&\quad = \sqrt{2(1-\cos\omega)} = 2\sin\frac{\omega}{2}
\end{aligned}$$

Moreover

$$\tan\theta = \sqrt{\frac{m\eta}{1-m\eta}}, \qquad \cot\theta = \sqrt{\frac{1-m\eta}{m\eta}}$$

We have finally

$$W = -\frac{4\pi\alpha}{\pi}\sin\frac{\omega}{2}\sqrt{\frac{1-m\eta}{m\eta}} - \frac{2\pi\alpha}{\pi}\tan^{-1}\left\{\frac{2\sin\frac{\omega}{2}\sqrt{m\eta(1-m\eta)}}{1-2m\eta\sin^2\frac{\omega}{2}}\right\}$$

APPENDIX III

As may be seen clearly from Fig. 3.2, the contributions to pressure coefficient along z axis due to waves A and A' are equal. By Eqn. (119), (3.5) (3.10), the c_p along z axis (i.e. y = o) may be written as

$$c_p = -\frac{2}{U}(2\,u_{I}) = k_p \frac{\alpha}{m}$$

where $$k_p \equiv \frac{8}{\pi}\Lambda$$

here Λ has the same meaning as in (3.5).

Let x = 2md, and then make all length dimension — less by dividing with d/2. We have

$$\Lambda = \tan^{-1}\left\{\frac{\sqrt{\eta^*(1-\eta^*)}\ \sin\frac{\omega}{2}}{\frac{1}{2} - \eta^*\sin^2\frac{\omega}{2}}\right\}$$

where

$$\eta^* \equiv m\frac{\varpi}{x} = \frac{1}{4}\sqrt{1+z^{*2}}$$

$$\omega = \tan^{-1} z^* < \frac{\pi}{2}$$

$$z^* \equiv \frac{z}{\left(\frac{d}{2}\right)}$$

Refer to Fig. 3.2,

for P_o : $z^* = 4$

for P_1 : $z^* = \sqrt{15} = 3.873$

for P_2 : $z^* = \sqrt{3} = 1.732$

Thus,

for $z^* = 4$ to $\sqrt{15}$, $k_p = 0$

for $z^* = \sqrt{15}$ to $\sqrt{3}$, k_p is calculated from the formula given above.

z^*	k_p
$\sqrt{15} = 3.873$	0
3.85	.8711
3.75	1.622
3.50	2.017
3.25	2.075
3.00	2.045
2.75	1.975
2.50	1.882
2.25	1.773
2.00	1.648
1.75	1.508
$\sqrt{3} = 1.732$	1.497

Coefficient k_p is plotted in Fig. 5, p. 51.

APPENDIX IV

Refer to Fig. 4.2. Contributions to pressure coefficient along z axis due to waves A and A' are equal and contribution due to waves α and α' are also equal. Thus along z axis (y = o), in region III'

$$c_p = -\frac{2}{U}\left(u_{II} + u_{II'} + \frac{2U\alpha}{m} \right) \equiv k_p \frac{\alpha}{m} \tag{1}$$

where $$k_p \equiv \frac{8}{\pi}\Lambda_1 - 4 \tag{2}$$

Here Λ_1 has the same meaning as Λ in (3.5), except with y = o

$$\varpi_1 = \sqrt{\left(-\frac{r}{2}\right)^2 + z^2} \tag{3}$$

$$\omega_1 = \tan^{-1}\left(\frac{z}{-\frac{r}{2}}\right) \tag{4}$$

Let $x = m\left(\frac{r}{2} + 3\frac{d}{2}\right)$ and make all length dimensionless by dividing with $\frac{d}{2}$ as in Appendix III. Take $\frac{r}{d} = 2.5$. Then

$$\Lambda_1 = \tan^{-1}\left\{ \frac{\sqrt{\eta_1^*(1-\eta_1^*)}\,\sin\frac{\omega_1}{2}}{\frac{1}{2} - \eta_1^*\sin^2\frac{\omega_1}{2}} \right\} \tag{5}$$

$$\eta_1^* \equiv m\frac{\varpi}{x} = \frac{1}{5.5}\sqrt{6.25 + z^{*2}} \tag{6}$$

$$\omega_1 = \tan^{-1}\left(-\frac{z^*}{2.5}\right), \qquad \frac{\pi}{2} < \omega_1 < \pi \tag{7}$$

For k_p along z axis in region III", we have, in addition to that given by (2), the contributions from waves α, α'. Denote the latter by $\hat{k}_p$,

$$\hat{k}_p = \frac{8}{\pi}\Lambda_2 \tag{8}$$

[illegible] as Λ in (3.5), except with y = o.

$$[\text{illegible}] \quad (9)$$

$$\omega_2 = [\text{illegible}] \left(\frac{z^*}{[\text{illegible}]} \right) \quad (10)$$

i.e.

$$\Lambda_2 = [\text{illegible}] \left\{ \frac{\sqrt{[\text{illegible}]\,(1-\eta_2^{*})}\,\sin\frac{\omega_2}{2}}{[\text{illegible}] - \eta_2^{*}\sin^2\frac{\omega_2}{2}} \right\} \quad (11)$$

$$\eta_2^{*} = [\text{illegible}] \sqrt{[\text{illegible}] + z^{*2}} \quad (12)$$

$$\omega_2 = \tan^{-1}\left(\frac{z^*}{[\text{illegible}]} \right), \qquad \omega_2 < \frac{\pi}{2} \quad (13)$$

Refer to Fig. 4.2

for P_0, $z^* = 5.5$

P_1, $z^* = \sqrt{24} = 4.8990$

P_2, $z^* = \sqrt{10} = 3.1623$

P_3, $z^* = \sqrt{6} = 2.4495$

Hence write $\quad c_p = k_p \frac{d}{m}$

(1) for $z^* = 5.5$ to $\sqrt{24}$ $\qquad k_p = 4$

(2) for $z^* = \sqrt{24}$ to $\sqrt{10}$

$$k_p = \frac{8}{\pi}\Lambda_1 - 4$$

(numerical values tabulated in next page)

z^*	k_p
$\sqrt{24}$ = 4.8990	4
4.85	3.1675
4.75	2.6061
4.50	1.9012
4.25	1.4927
4.00	1.2024
3.75	.9776
3.50	.7937
$\sqrt{10}$ = 3.1623	.5887

(3) for $z^* = \sqrt{10}$ to $\sqrt{6}$

$$k_p = \frac{8}{\pi}\Lambda_1 + \frac{8}{\pi}\Lambda_2 - 4$$

z^*	$\frac{8}{\pi}\Lambda_1 - 4$	$\frac{8}{\pi}\Lambda_2$	k_p
$\sqrt{10}$ = 3.1623	.5887	0	.5887
3.0	.5032	.2257	.7289
2.75	.3845	.3171	.7016
2.50	.2788	.3523	.6311
$\sqrt{6}$ = 2.4495	.2590	.3555	.6145

Coefficient k_p is plotted in Fig. 6, p. 52.

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